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HISTORY

Neugebauer, O. The Babylonian method for the computation of the last visibilities of Mercury. Proc. Amer. Philos. Soc. **95**, 110–116 (1951).

Neugebauer, O. A Greek table for the motion of the sun. Centaurus **1**, 266–270 (1951).

Kennedy, E. S. An Islamic computer for planetary latitudes. J. Amer. Oriental Soc. **71**, 13–21 (1951).

Rajagopal, C. T., and Vedamurthi Aiyar, T. V. On the Hindu proof of Gregory's series. Scripta Math. **17**, 65–74 (1951).

Frajese, Attilio. La geometria Greca e la continuità. Archimede **3**, 98–104 (1951).

Frajese, Attilio. Sur la signification des postulats euclidiens. Arch. Internat. Hist. Sci. (N.S.) **4**, 383–392 (1951).

Depman, I. Ya. A forgotten edition of Euclid's "Elements" in the Russian language. Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **3**, 467–474 (1950). (Russian)

Chakaya, D. G. The arithmetic and algebra of the Georgians and peoples of the Near East in the Georgian monuments of astronomical literature. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **17**, 315–340 (1949). (Georgian. Russian summary)

Zubov, V. P. The question of "indivisibles" and infinity in the old Russian literary monuments of the 15th century. Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **3**, 407–430 (1950). (Russian)

Molodšiĭ, V. N. The teaching of natural numbers in the 18th century. Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **3**, 431–466 (1950). (Russian)

Severi, Francesco. Divagazioni anche didattiche sulla matematica del novecento. Archimede **3**, 45–55 (1951).

van Dantzig, D. Some historical relations between mathematical and descriptive statistics. Statistica, Rijswijk **4**, 233–247 (1950). (Dutch. English summary)

Harkin, Duncan. The development of modern algebra. Norsk Mat. Tidsskr. **33**, 17–26 (1951).

Agostini, Amedeo. Il contributo Italiano al sorgere dell'analisi infinitesimale. Archimede **3**, 168–173 (1951).

Godeaux, Lucien. L'état actuel des recherches mathématiques en Belgique. Bull. Soc. Roy. Sci. Liège **19**, 514–524 (1950).

Szőkefalvi-Nagy, Béla. Résultats soviétiques dans le domaine de l'analyse fonctionnelle. Mat. Lapok **2**, 5–33 (1951). (Hungarian. French and Russian summaries)

Miranda, C. Risultati concernenti la risoluzione delle equazioni funzionali lineari dovuti all'Istituto Nazionale per le Applicazioni del Calcolo. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 346–353 (1950) = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **295** (1951).

Kramar, F. D. Questions of the foundations of analysis in the works of Wallis and Newton. Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **3**, 486–508 (1950). (Russian)

Coolidge, Julian L. Six female mathematicians. Scripta Math. **17**, 20–31 (1951).

The women referred to in the title are Hypatia (375–415 A.D.), Maria Gaetana Agnesi (1718–1799), Emilie, Marquise du Chatelet (1706–1749), Mary Somerville (1780–1872), Sophie Germain (1776–1831), and Sonya Kovalevsky (1850–1891).

Baltaga, V., Drinfel'd, G., and Levin, B. Naum Il'ič Ahiezer (for his fiftieth birthday). Uspehi Matem. Nauk (N.S.) **6**, no. 2(42), 191–194 (1 plate) (1951). (Russian)

Rybnikov, K. A. Viktor Viktorovič Bobynin. Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **3**, 343–357 (1 plate) (1950). (Russian)

The discussion of Bobynin's work is followed (pp. 358–396) by a list of his published works prepared by A. M. Lukomskaya.

Tambs-Lyche, R. On Harald Bohr's mathematical work. Norsk Mat. Tidsskr. **33**, 2–16 (1951). (Norwegian)

★v. David, L. Die beiden Bolyai. Elemente der Math. Beiheft no. 11. Verlag Birkhäuser, Basel, 1951. 24 pp. 3.50 Swiss francs.

★Winter, E. J. Leben und geistige Entwicklung des Sozialethikers und Mathematikers Bernard Bolzano, 1781–1848. Hallische Monographien no. 14, 100 pp. Max Niemeyer Verlag, Halle (Saale), 1949. 9 DM.

Fehr, H. Obituary: A. Buhl, 1878–1949. Enseignement Math. **39** (1942–1950), 6–8 (1951).

Tietze, Heinrich. Dem Andenken an C. Carathéodory. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1950**, 85–101 (1951).

Javillier, Maurice. Notice nécrologique sur Élie Cartan (1869–1951). C. R. Acad. Sci. Paris **232**, 1785–1791 (1951).

Carruccio, Ettore. La matematica nel pensiero di Cartesio. Rivista Mat. Univ. Parma **2**, 133–152 (1951).

***Čebotarev, N. G. Sobranie sočinenii. [Collected Works]. vol. 3.** Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1950. 171 pp.

The first two volumes were published in 1949 [cf. these Rev. **11**, 572, 872]. The present volume contains two further papers [Comment. Math. Helv. **6**, 235–283 (1934); Newton's polygon and its role in the contemporary development of mathematics, from "Isaac Newton", Izdat. Akad. Nauk SSSR, 1943, pp. 99–126], his Mathematical Autobiography [Uspehi Matem. Nauk (N.S.) **3**, no. 4(26), 3–66 (1948); these Rev. **10**, 174], a list of his published works, and an essay by I. P. Šafarevič, "On N. G. Čebotarev's work, 'The determination of the density of a set of prime numbers belonging to a given substitution class' " [Math. Ann. **95**, 191–228 (1925)].

***Itard, Jean. Pierre Fermat.** Elemente der Math. Beiheft no. 10. Verlag Birkhäuser, Basel, 1950. 24 pp. 3.50 Swiss francs.

Julia, Gaston. La vie et l'œuvre de J.-L. Lagrange. Enseignement Math. **39** (1942–1950), 9–21 (1951).

Lyusternik, L. A., and Šura-Bura, M. R. Mihail Alekseevič Lavrent'ev (for his 50th birthday). Uspehi Matem. Nauk (N.S.) **6**, no. 1(41), 190–192 (1 plate) (1951). (Russian)

Rybkin, G. F. Materialism—the main line of N. I. Lobačevskiĭ's philosophy. Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **3**, 9–29 (1950). (Russian)

Yanovskaya, S. A. On the philosophy of N. I. Lobačevskiĭ. Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **3**, 30–75 (1950). (Russian)

Nagaeva, V. M. The pedagogical views and activity of N. I. Lobačevskiĭ. Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **3**, 76–153 (1950). (Russian)

Bespamyatnyh, N. D. On the theory of negative numbers in Lobačevskiĭ's work. Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **3**, 154–170 (1950). (Russian)

Bronšteĭn, I. N. On the history of N. I. Lobačevskiĭ's "Survey of Instruction in Pure Mathematics." Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **3**, 171–194 (1950). (Russian)

***Lobačevski, N. I. Geometriska ispitivanja iz teorije paralelnih linija. [Geometric Investigations on the Theory of Parallel Lines]. Translated and annotated by Branislav Petronijević.** 2d ed. Srpska Akademija Nauka. Klasični Spisi, Knj. III. Matematički Institut, Knj. 3. Belgrade, 1951. v+83 pp. (Serbo-Croatian)

Translation of Lobačevskiĭ's Geometrische Untersuchungen zur Theorie der Parallellinien [Berlin, 1840]. The first edition of this translation appeared in Belgrade in 1914.

***Rossinskiĭ, S. D. Boleslav Kornelievič Mlodzeevskiĭ, 1858–1923.** Izdat. Moskov. Univ., Moscow, 1950. 52 pp. (5 plates).

***Taton, René. Gaspard Monge.** Elemente der Math. Beiheft no. 9. Verlag Birkhäuser, Basel, 1950. 24 pp. 3.50 Swiss francs.

Taton, René. Remarques sur la diffusion des théories mathématiques de Monge. Thalès **5** (1948), 43–49 (1949).

Nikolaĭ Ivanovič Mushelišvili. For his sixtieth birthday. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 265–278 (1 plate) (1951). (Russian)

A survey of his work and a list of his published papers.

Keldyš, M. V., and Sobolev, S. L. Nikolaĭ Ivanovič Mushelišvili (for his sixtieth birthday). Uspehi Matem. Nauk (N.S.) **6**, no. 2(42), 185–190 (1 plate) (1951). (Russian)

Maron, I. A. Academician M. V. Ostrogradskiĭ as an organizer of instruction in mathematical sciences in the military schools of Russia. Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **3**, 197–340 (1 plate) (1950). (Russian)

Koutský, Karel. Obituary: Karel Petr. Časopis Pěst. Mat. Fys. **75**, 341–345 (1 plate) (1950). (Czech)

Sobolev, S. L. On the fiftieth birthday of Ivan Georgievič Petrovskiĭ. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 201–204 (1 plate) (1951). (Russian)

A list of Petrovskiĭ's published papers is included.

Lorch, E. R. Obituary: Joseph Fels Ritt. Bull. Amer. Math. Soc. **57**, 307–318 (1951).

A list of Ritt's published mathematical works is included.

Markuševič, A. I. The contribution of Yu. V. Sohockiĭ to the general theory of analytic functions. Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **3**, 399–406 (1 plate) (1950). (Russian)

Agostini, Amedeo. I baricentri trovati da Torricelli. Boll. Un. Mat. Ital. (3) **6**, 149–159 (1951).

Patroni, Adriano. Il manoscritto M 39 r dei manoscritti di Leonardo da Vinci. (Raccolta di Francia). Boll. Un. Mat. Ital. (3) **6**, 159–162 (1951).

Schmetterer, Leopold. Obituary: Abraham Wald. Statist. Vierteljschr. **4**, 69–74 (1951).

A list of Wald's published work is included.

Fehr, Henri. Obituary: Rolin Wavre, 1896–1949. Actes Soc. Helv. Sci. Nat. **130**, 420–428 (1 plate) (1950).

A list of Wavre's published papers is included.

FOUNDATIONS

Mostowski, Andrzej. On the rules of proof in the pure functional calculus of the first order. J. Symbolic Logic 16, 107–111 (1951).

In the pure functional calculus of first order F_p^1, a formula is provable if and only if it is valid in every nonempty set. Let F_p^* be the system in which all of Church's axioms for F_p^1, and also the rules II, II′, II″, II‴, III hold, but in which the rule of modus ponens is replaced by the following weaker rule: If A and B are wffs such that all individual variables free in a are free in B, and, if A and $A\supset B$ are theorems, then so is B. It is then shown that a formula is provable in F_p^* if and only if it is valid in every set.

I. L. Novak (Princeton, N. J.).

Łukasiewicz, Jan. On variable functors of propositional arguments. Proc. Roy. Irish Acad. Sect. A. 54, 25–35 (1951).

The author considers the modifications in his conception of propositional algebra when a unary functor δ with a rule permitting the substitution for $\delta\alpha$ ("α" is a syntactical variable for any formula of the algebra) of an arbitrary functino of α. Such functors, along with quantifiers binding propositional variables, were introduced by Leśniewski to form his "protothetic". The author observes that such a rule of substitution requires some supplementary formulation in order to fit into his technique; he does this using an apostrophe for what would be, in the language of functional abstraction, the bound variable of the function to be substituted. With the resulting technique he can represent "theses" valid for an arbitrary function of one argument. He shows that the equivalence of P and Q can be stated in the form $C\delta P\delta Q$ and that this form of equivalence can be used in making definitions without the necessity of postulating a special sort of "identity by definition". (In the theories of Leśniewski and Łukasiewicz the symbols appearing in the theses are the objects being talked about, and hence such an identity by definition would have to be regarded as an additional idea to be formalized.) In conclusion the author shows that every thesis of the classical algebra can be deduced from $C\delta COOC\delta O\delta p$ by a formalization of the method of truth tables. The paper mentions certain generalizations and contains historical remarks showing that the author was influenced by Leśniewski, Sobociński, and C. A. Meredith [see the following review]. *H. B. Curry.*

Meredith, C. A. On an extended system of the propositional calculus. Proc. Roy. Irish Acad. Sect. A. 54, 37–47 (1951).

The author considers the completeness of various formulations of protothetic [see the preceding review]. These include not only a form with a constant O, propositional variables, the constant functor C (implication), and a single variable functor δ (as in the preceding review), but also a form containing several variable functors and a universal quantifier Π which may bind either propositional or functorial variables. In the first case he shows that the single axiom $C\delta\delta O\delta p$ suffices for the system, in the sense that, given any formula, either it can be deduced from the axiom by the rules of substitution and modus ponens, or an arbitrary formula can be deduced from it; moreover the alternatives are decidable. In the second case he assumes the same axiom, with O defined as Πpp, and also the rules for inserting a universal quantifier in premise and conclusion of an implication. He gives a method of reduction for this system (depending on the fact that the quantification is over a finite range), and claims that the completeness holds for this system also. The formal developments are very tedious.

H. B. Curry (State College, Pa.).

Vaccarino, Giuseppe. Il calcolo dei predicati. Archimede 3, 105–111 (1951).

McNaughton, Robert. A theorem about infinite-valued sentential logic. J. Symbolic Logic 16, 1–13 (1951).

The main results of this paper are two theorems about a logic with truth-values ranging over all the real numbers x such that $0\leqq x\leqq 1$. The highest truth-value is 1 and the truth-values of the primitive functions, implication and negation, are determined as follows. If p, q have the truth-values x, y respectively then $p\supset q$, $\sim p$ have the truth-values $\min(1-x+y, 1)$, $1-x$ respectively.

The author first shows that for any given integers $b, m_1, \cdots, m_n$ there is a logical formula $S(p_1, \cdots, p_n)$ such that whenever $p_1, \cdots, p_n$ take the truth-values $x_1, \cdots, x_n$, respectively, $S(p_1, \cdots, p_n)$ takes the truth-value

$$\min(\max(0, b+m_1x_1+\cdots+m_nx_n), 1).$$

From this he deduces the more general theorem that necessary and sufficient conditions for a function $S(p_1, \cdots, p_n)$ which always takes the truth-value $f(x_1, \cdots, x_n)$ to be definable in terms of the primitives are: (i) f is single-valued and continuous and $0\leqq f(x_1, \cdots, x_n)\leqq 1$ where $0\leqq x_i\leqq 1$, $i=1, \cdots, n$; (ii) there are a finite number of distinct polynomials $\lambda_1, \cdots, \lambda_\mu$, each $\lambda_j=b_j+m_{1j}x_1+\cdots+m_{nj}x_n$ where all the b's and m's are integers, such that for every $(x_1, \cdots, x_n)$, $0\leqq x_i\leqq 1$, $i=1, \cdots, n$, there is a j, $1\leqq j\leqq \mu$, such that $f(x_1, \cdots, x_n)=\lambda_j(x_1, \cdots, x_n)$.

He then discusses the relationship between this logic and the $\aleph_0$-valued logic of Łukasiewicz and Tarski [Soc. Sci. Lett. Varsovie. C. R. Cl. III, Sci. Math. Phys. 23, 30–50 (1930)], showing that if the function $f(x_1, \cdots, x_n)$ is defined for all $(x_1, \cdots, x_n)$ such that x_i is rational and $0\leqq x_i\leqq 1$, $i=1, \cdots, n$, a necessary and sufficient condition that the formula $S(p_1, \cdots, p_n)$ of the $\aleph_0$-valued system always takes the truth-value $f(x_1, \cdots, x_n)$ is that for every $(\xi_1, \cdots, \xi_n)$ such that $0\leqq \xi_i\leqq 1$, f has a unique limiting value at $(\xi_1, \cdots, \xi_n)$ equal to the truth-value of $S(p_1, \cdots, p_n)$ in the original system. He then deduces that in both infinite valued systems necessary and sufficient conditions that there exists a formula whose minimum and maximum truth-values are a, b respectively are $0\leqq a\leqq b\leqq 1$, $a=0$ or $b=1$ (or both), and a and b are rational.

The paper concludes with a discussion of the relationship to the finite valued logics of Łukasiewicz [op. cit.]. It is shown that if the formula $S(p_1, \cdots, p_n)$ of the m-valued system takes the truth-value

$$\bar{f}(s_1/(m-1), \cdots, s_n/(m-1))=s/(m-1)$$

when p_i takes the truth-value $s_i/(m-1)$, $i=1, \cdots, n$, then a necessary and sufficient condition that it be definable in terms of the primitives is that for each $(s_1/(m-1), \cdots, s_n/(m-1))$ if d is a common divisor of $s_1, \cdots, s_n, m-1$ then d divides s.

Errata: p. 5, l. 30 should be omitted; p. 5, l. 33 should precede l. 31 and l. 32. *A. Rose* (Aberdeen).

Schütte, Kurt. Die Eliminierbarkeit des bestimmten Artikels in Kodifikaten der Analysis. Math. Ann. 123, 166–186 (1951).

Given a formalization of analysis which may contain bound variables for numbers, functions and predicates, and in which there is an ι-symbol. Identity is assumed to be one of the predicates. (The terms above are used in essentially the same way as in Hilbert and Bernays, Grundlagen der Mathematik, vol. I, p. 451 [Springer, Berlin, 1934].) The axioms include the usual ones for quantification theory for all three types of variables, the necessary identity axioms and rules of inference, and various mathematical formulas which may be chosen arbitrarily. In addition, it must contain the following axiom of choice:

$$(x_1)(\cdots)(x_n)(\exists y)\mathfrak{A}(x_1, \cdots, x_n, y)\rightarrow (\exists u^n)(x_1\cdots x_n)\mathfrak{A}(x_1, \cdots, x_n, u^n(x_1, \cdots, x_n)),$$

where $x_1, \cdots, x_n$ are numbers, $\mathfrak{A}$ is a predicate and u^n is an n-valued function. It is shown that for any formula $\mathfrak{B}$ of this system there is a proof which contains no more superpositions of ι's than there are in $\mathfrak{B}$. Hence the ι-symbol can be eliminated in such a system. *I. L. Novak.*

Kuroda, Sigekatu. Intuitionistische Untersuchungen der formalistischen Logik. Nagoya Math. J. 2, 35–47 (1951).

Main result: Let A be a provable formula in the ordinary predicate calculus and A' the formula which results if a double negation is placed before A and before every part of A consisting of an all-operator with its operand; then A' is provable in the intuitionistic predicate calculus. The author suggests an application to a proof of consistency for classical number theory relatively to intuitionistic number theory, but presumably the results obtainable by his method will not exceed those of Gödel [Ergebn. Math. Kolloqu. 4, 34–38 (1933)]. *A. Heyting* (Amsterdam).

Markov, A. The impossibility of algorithms for the recognition of certain properties of associative systems. Doklady Akad. Nauk SSSR (N.S.) 77, 953–956 (1951). (Russian)

A general theorem is announced, with an outline proof, from which the results of a previous paper [same vol., 19–20 (1951); these Rev. 12, 661; called M1 in this review] follow as special cases. Let "K-system" mean "associative system with a finite set of generating relations". A K-system, S, is said to be included in another, T, if S is isomorphic to a subsystem of T. A property P of K-systems is invariant if it is preserved under isomorphisms. As in the cited review, we denote by $[R, A]$ the K-system defined by the relations R in the alphabet A. The "recognition problem" for a property P and an alphabet A is the problem of finding an algorithm for deciding, for any finite set of relations R in A, whether or not $[R, A]$ has the property P. Theorem 1. Let P be an invariant property of K-systems. If there exist both a K-system having the property, and a K-system not included in any K-system having the property, then there exists an alphabet for which the recognition problem for P is unsolvable. If a K-system with the property P is definable with n letters, the recognition problem for P is unsolvable for alphabets of $\geqq m+4$ letters.

The method of proof is a generalisation of that used in M1, the relations D and alphabet $B=\{a, b, c, d\}$ of that paper being replaced by $R_3\cup R_4$ and $B\cup B_4$ respectively, where R_3, R_4 and B_4 are as follows. $[R_3, \{a, b\}]$ is a K-system that includes (in the above sense) the "direct product" of a K-system S_1 having an unsolvable word-problem, and a K-system S_0 not included in any having property P. (Here the "direct product" of $[R, A]$ and $[R', A']$ is $[R\cup R', A\cup A']$, provided that $A\cap A'=\phi$.) The system $[R_4, B_4]$ is a K-system having property P, such that $B\cap B_4=\phi$. An immediate corollary of the theorem is that if P is a hereditary property, i.e., holds for all subsystems of K-systems that have the property, and if there exist K-systems that have, and K-systems that have not the property, then the recognition problem is unsolvable for alphabets of 4 or more letters. A deduction from the main theorem not included in M1 is that there is no algorithm for deciding if a given K-system is finite. *M. H. A. Newman* (Manchester).

★Schmidt, Arnold. Mathematische Grundlagenforschung. Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. I 1, 2. Band I. Algebra und Zahlentheorie. Teil 1. A. Grundlagen. B. Algebra. Heft 1. Teil II. B. G. Teubner, Leipzig, 1950. 48 pp. $1.34.

This pamphlet contains a brief survey of some aspects of work in the foundations of mathematics. It was written in 1939 but has to some extent been brought up to date by additional footnotes. The first section sketches the program of Hilbert's Beweistheorie and Gödel's arithmetization of syntax and incompleteness theorem, as well as some basic concepts of semantics. The second section is devoted to the formalization of elementary number theory and to the possibility of proving it consistent. The logical paradoxes and the doctrine of types are discussed in the third section, and the final section is devoted to a sketch of the basic ideas of intuitionism. *R. M. Martin* (Philadelphia, Pa.).

Fitch, Frederic B. A demonstrably consistent mathematics. II. J. Symbolic Logic 16, 121–124 (1951).

This paper is concerned with the logistic system K', developed by the author in several previous papers [J. Symbolic Logic, especially 13, 95–106 (1948); 14, 9–15 (1949); these Rev. 9, 559; 10, 669]. Some elementary properties of continuous functions are defined. It is then shown how, in particular, utilizing only the procedures of K', one can prove that if a real function is continuous in an interval (and is appropriately expressible in K') then the class of values of the function at points in the interval has a least upper bound expressible in K'. Interest attaches to the methods employed because of the available consistency proof for K'.

R. M. Martin (Philadelphia, Pa.).

Neves Real, Luís. Kurt Gödel and the problems of the foundations of mathematics and the theory of sets. Gaz. Mat., Lisboa 12, no. 48, 1–8 (1951). (Portuguese)

Expository paper.

Popadić, Milan. Induction complète. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Éd. Spéc. 2, 29 pp. (1950). (Serbo-Croatian. Russian and French summaries)

Cet article est de la nature instructive et informative et s'occupe de la méthode de demonstration nommée. . . . C'est un abrégé de différentes définitions du principe de l'induction complète, avec plusieurs exemples de mathématiques élémentaires principalement. À la fin, après un aperçu historique, on a indiqué l'importance de cette méthode.

Résumé de l'auteur.

Pi Calleja, Pedro. Grandjot's objection to Peano's theory of natural numbers. Math. Notae 9, 143–151 (1949). (Spanish)

It is well known that the recursive "definitions" of addition and multiplication on the basis of the Peano postulates for the natural numbers, do not constitute definitions in the ordinary sense, but have either to be taken as new postulates or to be supplemented by a more or less lengthy argument involving an underlying logic stronger than the classical predicate calculus of first order. This was the basis of an objection by Grandjot which was mentioned in the preface to Landau's "Grundlagen der Analysis" [Akademische Verlagsgesellschaft, Leipzig, 1930]. The present paper is a discussion of this objection, of the argument given by Landau (credited to Kalmár), its relation to that of Dedekind, etc.

H. B. Curry (State College, Pa.).

Pi Calleja, Pedro. The included middle in Russell's counterparadox. Math. Notae 9, 152–154 (1949). (Spanish)

Remarks on the self-cited catalog form of the Russell paradox [cf. F. Gonseth, Les mathématiques et la realité . . . , Alcan, Paris, 1936]. *H. B. Curry* (State College, Pa.).

Levi, B. Concerning the note of Dr. Pi Calleja. On logical paradoxes and the principle of tertium non datur. Math. Notae 9, 155–159 (1949). (Spanish)

Philosophical remarks about the paradoxes.

H. B. Curry (State College, Pa.).

***Carnap, Rudolf. The Nature and Application of Inductive Logic. Consisting of Six Sections from Logical Foundations of Probability.** The University of Chicago Press, Chicago, Ill., 1951. viii+80 pp. (paged 161–202; 242–279). $1.25.

This is a reprint of §§41–43 and 49–51 from chapter IV of the author's book, Logical Foundations of Probability [Univ. of Chicago Press, 1950; these Rev. 12, 664].

ALGEBRA

Armsen, Paul. Eine Bemerkung über Inversionen in Permutationen. Norske Vid. Selsk. Forh., Trondheim 23, 87–90 (1951).

The author gives a combinatorial proof of a recurrence formula, due to E. Jacobsthal [same Forh. 22, no. 11, 37–41 (1950); these Rev. 11, 574] for the number of permutations of n elements with r inversions. *J. Riordan.*

Hanani, Haim. On the number of straight lines determined by n points. Riveon Lematematika 5, 10–11 (1951). (Hebrew. English summary)

The problem is that of determining the least number p of straight lines connecting n points so that every pair of points is connected. The author states that aside from the trivial case of all points collinear, $p>n$ except when (i) $n-1$ points are collinear and $p=n$, (ii) $n=m^2+m+1$, $m=1, 2, \cdots$ and $p=n$. The proof and statement of the last case are not clear.

J. Riordan (New York, N. Y.).

***Schreier, O., and Sperner, E. Introduction to Modern Algebra and Matrix Theory.** Translated by Martin Davis and Melvin Hausner. Chelsea Publishing Company, New York, N. Y., 1951. viii+378 pp. $4.95.

A translation of the two volumes of Einführung in die analytische Geometrie und Algebra [Hamburger Math. Einzelschr., Hefte 10, 19, Teubner, Leipzig, 1931, 1935] except for the last chapter of vol. II on projective geomety.

Goddard, L. S. On positive definite quadratic forms. Publ. Math. Debrecen 2, 46–47 (1951).

This paper is essentially Kronecker's derivation in matric notation, of the well-known criterion that a quadratic form be definite. *B. W. Jones* (Boulder, Colo.).

Parodi, Maurice. Sur la formation de matrices définies positives. C. R. Acad. Sci. Paris 232, 2390–2392 (1951).

Let A be a symmetric, positive definite square matrix of order n with real elements (a_{ij}). Let $\alpha_{ij}^{(n)}, \alpha_{ij}^{(n-1)}, \cdots, \alpha_{ij}^{(1)}$ be the elements of the inverse of A and of a chain of principal minors of A. It is shown that a varied matrix $(a_{ij}+\eta_{ij})$ is positive definite whenever $\sum_{ij}|\eta_{ij}|^2<[\sum_{ij}|\alpha_{ij}^{(n)}|^2]^{-1}$. The proof depends directly on related results of Ostrowski, including an unpublished result that $\sum_{ij}|\alpha_{ij}^{(p)}|^2$ increases with p. *G. E. Forsythe* (Los Angeles, Calif.).

Landsberg, P. T. On matrices whose eigenvalues are in arithmetic progression. Proc. Cambridge Philos. Soc. 47, 585–590 (1951).

Necessary and sufficient conditions that a Hermitian matrix W have eigenvalues in arithmetic progression are given; the matrices for a one-dimensional harmonic oscillator and a Cartesian component of angular momentum are treated as special cases. Further properties of two matrices A, B are deduced (for the oscillator, A, B are proportional to the position and momentum operators). *T. E. Hull.*

Richter, Hans. Zum Logarithmus einer Matrix. Arch. Math. 2, 360–363 (1950).

Let A be a matrix of order n with complex elements none of whose latent roots λ_i are zero or negative real. If the λ_i are distinct and

$$A=C\begin{bmatrix}\lambda_1 \cdots 0\\ \cdot \quad \cdot \quad \cdot\\ 0 \cdots \lambda_n\end{bmatrix}C^{-1},$$

the logarithm of A is defined by

$$\ln A=C\begin{bmatrix}\ln\lambda_1 \cdots & 0\\ \vdots \quad \vdots & \vdots\\ 0 \quad \cdots & \ln\lambda_n\end{bmatrix}C^{-1}.$$

If there are repetitions amongst the λ_i then

$$\ln A=\lim_{t\to 0}\ln A(t),$$

where $A(t)$ has distinct latent roots and where

$$\lim_{t\to 0}\ln A(t)=A.$$

The results of a previous paper [Math. Ann. 122, 16–34 (1950); these Rev. 12, 235] are used by the author to express $\ln A$ in the form $\sum_{\nu=0}^{n-1}c_\nu A^\nu$, in which the coefficients c_ν can be evaluated from the characteristic function

$$\phi(z)\equiv|zE-A|\equiv\sum_{\nu=0}^{n-1}a_\nu z^{n-\nu},$$

without determining the λ_i. If $\{Q\}$ denotes the trace of the

matrix Q and if $\bar{Q}=Q-(1/n)\{Q\}E$, E being the unit matrix, then the result is as follows:

$$\ln A=\frac{\ln|A|+2\pi li}{n}\cdot E+\sum_{k=1}^{n-1}c_k\cdot\bar{A}^k,$$

where l ($l=0$ if A is real) is the number of downward crossings of the negative real axis made by $(-t)^n\phi((t-1)/t)$ as t increases from 0 to 1, where $c_k=\int_{-\infty}^{0}\{g_k(u)/\phi(u)\}du$ and $g_k(u)=\sum_{r=0}^{n-1-k}a_{n-1-k-r}u^r$. *D. E. Rutherford.*

Jarušek, Jaroslav. Semiinvariants of ternary forms. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **47** (1946), 1–4 (1950).

This paper is a sketch of a systematic method for constructing the complete system of irreducible invariants, covariants, contravariants and mixed concomitants of a system of ternary forms by reducing the problem to that of a suitably chosen set of binary forms. The work is closely connected with an earlier paper by the same author [Rozpravy II. Třidy České Akad. **56**, no. 1 (1946); these Rev. **12**, 154]. A typical mixed ternary form F is of degree n in the set x_1, x_2, x_3 and r in a contragredient set u_1, u_2, u_3: it is regarded as a sum of double binary forms in x_2, x_3 and u_2, u_3, one form for each term when arranged in powers of x_1 and u_1. A device is employed for replacing double binary by simple binary forms; and from the knowledge of the complete system of a given set of binary forms that of the originating ternary forms is inferred by an extensional method. This last depends upon the ternary semiinvariants (given by the cofactor of the highest power of x_1 and u_1 in the ternary concomitant) and expressions derived therefrom by utilizing certain operators Δ_{ij} ($i, j=1, 2, 3, i\neq j$) that generalize on the Sylvester annihilator Ω of classical binary theory. *H. W. Turnbull* (Millom).

Duparc, H. J. A. On canonical forms. Math. Centrum Amsterdam. Rapport **ZW-1950-020**, 8 pp. (1950). (Dutch)

This is an account of the Lasker-Wakeford method of establishing the existence of a canonical form for a homogeneous polynomial $f(x)$ of order n in p variables x_i, together with an extension of the method to systems of several such forms. The method depends upon a correlative polynomial $\phi(u)$, homogeneous in p variables u_i which are contragredient to the x_i. Two such forms $\phi(u)$ and $\psi(x)$ are apolar when their lineo-linear invariant vanishes; e.g., when $(\sum_{i=1}^{p}(\partial/\partial u_i)(\partial/\partial x_i))^q\phi(u)\psi(x)=0$ if each of ϕ and ψ is of order q. Apolarity is easily extended to the case when ϕ and ψ have different orders.

The method arose through the discovery that merely counting of constants was an insufficient test for justifying a prescribed canonical form: thus a ternary quartic (with its $15=5\times3$ general coefficients) cannot be expressed as a sum of five fourth powers of linear forms (each with 3 coefficients). The Lasker-Wakeford theorem asserts that a legitimate canonical form for $f(x)$, which has, say, N parameters, is $F(X, m)$, which contains at least N parameters among its k variables X_j (each of which is a prescribed type of polynomial, linear for preference, in the x_i) together with its r explicit parameters m_k, provided that there is not a correlative form $\phi(u)$ dual to $f(x)$ and simultaneously apolar to every derivative of the types $\partial F/\partial X_j$ and $\partial F/\partial m_k$. The author extends this to a pencil of such forms $\lambda_1F(X, m)+\lambda_2G(X, m)$, and illustrates it with two ternary quadratics. [See Lasker, Math. Ann. **58**, 434–440 (1904); Wakeford, Proc. London Math. Soc. (2) **18**, 403–410 (1920).] *H. W. Turnbull.*

Turnbull, H. W., and Wallace, A. H. Clebsch-Aronhold symbols and the theory of symmetric functions. Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 155–173 (1951).

This paper does not claim to present new results but obtains old ones in a novel way. The classical Clebsch-Aronhold symbolism as applied to a bilinear form is interpreted in terms of the elements of a matrix A, in particular with regard to the symmetric functions of the characteristic roots of A. The various relations between these symmetric functions are shown to be obtainable from the first two fundamental theorems of classical invariant theory.

G. de B. Robinson (Toronto, Ont.).

Newell, M. J. On the multiplication of S-functions. Proc. London Math. Soc. (2) **53**, 356–362 (1951).

The author defines operators $\xi_1, \xi_2, \cdots, \xi_k$ which have the effect of reducing by unity the corresponding terms of a set of k suffixes $l_1, l_2, \cdots, l_k$. Alternant and symmetric functions of these operators are considered. The operators are used to give a new proof of Kostka's theorem concerning bialternants, monomial symmetric functions and products of homogeneous product sums. If the S-function $\{\lambda\}$ of $\alpha_1, \alpha_2, \cdots, \alpha_n$ is expressed in the form

$$A_0+A_1\alpha_1+A_2\alpha_1^2+\cdots+A_{\lambda_1}\alpha_1^{\lambda_1}$$

the operators are used to express A_i in terms of S-functions of $\alpha_2, \cdots, \alpha_n$. More generally a formula is obtained expressing $\{\lambda\}$ in the form $\sum T_{(\theta)}A_{(\theta)}$ where $T_{(\theta)}$ is a monomial symmetric function of $\alpha_1, \cdots, \alpha_p$ and $A_{(\theta)}$ is expressed in S-functions of $\alpha_{p+1}, \cdots, \alpha_n$. Expressing $T_{(\theta)}$ in terms of S-functions of $\alpha_1, \cdots, \alpha_p$ a proof is obtained of Murnaghan's rule for obtaining the product of two S-functions.

D. E. Littlewood (Bangor).

Garrett, James Richard. Normal equations and resolvents in fields of characteristic p. Duke Math. J. **18**, 373–384 (1951).

The author discusses the reduction of a general quintic to a normal form, when the coefficients lie in a field of characteristic p. He remarks that the method of reduction usually developed for fields of characteristic zero is still valid when the characteristic is greater than 5. It therefore remains only to settle the cases $p=2$, 3 and 5. This is done in the present paper. The reduction is effected by means of one or more Tschirnhausen transformations $y=cx^4+dx^3+ex^2+fx+g$, replacing the original unknown x by a new unknown y. When $p=2$ or 3, the normal form $y^5+a_4y+a_5$ is attained, while in the case $p=5$ the normal form $y^5+a_3y^2+a_5$ is used. In addition, an explicit formula for a sextic resolvent is obtained in each case.

W. Ledermann (Manchester).

Abstract Algebra

Croisot, Robert. Sous-treillis, produits cardinaux et treillis homomorphes de treillis semi-modulaires. C. R. Acad. Sci. Paris **232**, 27–29 (1951).

The author notes that all the conditions for semi-modularity discussed in his previous notes [same C. R. **231**, 12–14, 1399–1401 (1950); these Rev. **12**, 4, 473] are preserved in

passing from a lattice to a convex sublattice, and the majority are preserved under the formation of cardinal products. In particular, he solves Problem 46 of the reviewer's "Lattice Theory", 2d ed. [Amer. Math. Soc. Colloquium Publ., vol. 25, New York, 1948; these Rev. 10, 673]. Proofs are sketched. *G. Birkhoff* (Cambridge, Mass.).

***Kořínek, Vladimír. Le théorème de Jordan-Hölder dans les treillis.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 155–157. Centre National de la Recherche Scientifique, Paris, 1950.

A survey of results concerning the Jordan-Hölder and Schreier theorems in lattices based upon papers of the author [Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodvěd. 1941, no. 14 (1941); Rozpravy II. Třídy České Akad. 59, no. 23 (1949); these Rev. 7, 509; 12, 667]. *O. Borůvka* (Brno).

Moreira Gomes, Alvercio. Decomposition of partially ordered systems. Revista Cientifica 1, no. 2, 12–18 (1950). (English. Portuguese summary)

A direct elementary proof for the case $k=2$ of a theorem of Dilworth [Ann. of Math. (2) 51, 161–166 (1950); these Rev. 11, 309]. *G. Birkhoff* (Cambridge, Mass.).

Rennie, B. C. Lattices. Proc. London Math. Soc. (2) 52, 386–400 (1951).

The author defines a new topology (the "L-topology") in a lattice L, by taking as a basis of open sets, those convex sets S, whose intersection with any chain is an open set of the chain. He shows that if L is conditionally complete, star-convergence implies L-convergence. In a metric lattice, vx is continuous in the L-topology if and only if it is continuous on all chains; in this case ("continuous metric lattice"), metric convergence is equivalent to L-convergence. In the lattice $L(H)$ of closed subsets of a Hausdorff space, the empty set 0 is isolated if and only if H is compact; moreover, if H is locally compact, then L is a Hausdorff space in the L-topology. In any Banach lattice L^p, $p>1$, the L-topology is the metric topology; in any Banach lattice B, it is the star topology. Other theorems are proved and 7 unsolved problems stated. *G. Birkhoff.*

Jordan, Pascual. Über polynomiale Fastringe. Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. 1950, 337–340 (1951).

A Fastring satisfies the usual postulates for a ring except that addition need not be commutative, and the only distributive law assumed is that $(a+b)c=ac+bc$. The set of all single-valued transformations of a group into itself is one example. The author gives a number of other examples, and makes a few elementary observations about such systems. *N. H. McCoy* (Northampton, Mass.).

Bourne, Samuel. The Jacobson radical of a semiring. Proc. Nat. Acad. Sci. U. S. A. 37, 163–170 (1951).

A semiring is a set which is a semigroup relative to addition, a semigroup relative to multiplication, and both distributive laws hold. A right ideal I in a semiring S is said to be right semiregular if for every pair of elements i_1, i_2 in I there exist elements j_1, j_2 in I such that $i_1+j_1+i_1j_1+i_2j_2=i_2+j_2+i_1j_2+i_2j_1$. The sum of all right semiregular right ideals of S is a right semiregular two-sided ideal of S which is defined to be the right Jacobson radical of S. The left Jacobson radical is defined analogously, and it is shown that the right and left Jacobson radicals coincide, thus making it possible to refer simply to the Jacobson radical. If R is the Jacobson radical of the semiring S, then $S-R$ has zero Jacobson radical, R contains every nilpotent right or left ideal of S, the Jacobson radical of the matrix semiring S_n is R_n in case S has a unit element. The proofs of these and other related results involve extensive elementwise calculations. These results are generalizations of well known results of Jacobson [Amer. J. Math. 67, 300–320 (1945); these Rev. 7, 2]. *N. H. McCoy.*

Amitsur, Shimshon. Semi-group rings. Riveon Lematematika 5, 5–9 (1951). (Hebrew. English summary)

Similarly to the definition of group-rings, we define semi-group rings. We show as examples that polynomial rings and matrix rings are semi-group rings. *Author's summary.*

Nagata, Masayoshi. On the structure of complete local rings. Nagoya Math. J. 1, 63–70 (1950).

A local ring in this paper means a commutative ring R with identity in which the nonunits form an ideal M such that $\bigcap_{i=1}^{\infty} M^i=(0)$. (The customary definition requires that R be Noetherian.) The main result of this paper is: A complete local ring R contains a subring R_0 which maps modulo M onto all of R/M, and which is either a field, or a complete local ring whose maximal ideal is generated by a prime number p. For Noetherian local rings this has been proved by the reviewer [Trans. Amer. Math. Soc. 59, 54–106 (1946); these Rev. 7, 509], and the two proofs are essentially the same when R/M is of characteristic 0. When R/M is perfect and of characteristic $P\neq 0$, the present proof is considerably simpler than that of the reviewer. When R/M is imperfect, the present proof seems incorrect. It is based on the statement that a p-basis of R/M is a transcendence basis for R/M over its maximal perfect subfield; however, this statement is false. In an appendix the author gives an example of a local ring which is not Noetherian but in which M is finitely generated; this settles a question left open by the reviewer. *I. S. Cohen* (Cambridge, Mass.).

Zelinsky, Daniel. Complete fields from local rings. Proc. Nat. Acad. Sci. U. S. A. 37, 379–381 (1951).

A class of complete topological fields is constructed which differs from the fields complete in valuations, the only ones known so far. Consider a field F, the quotient field of a ring R, and topologize (R-topology) both F and R by declaring the nonzero ideals of R to be a system of neighbourhoods of zero. It is first shown that a complete local ring R is also complete in the R-topology. Next it is proved that a complete local ring without zero division has a quotient field complete in the R-topology. Finally, it is shown that most of the resultant fields have no valuation preserving their topologies. A concrete example is the quotient field of the ring of formal power series in n variables ($n\geqq 2$). *O. Todd-Taussky* (Washington, D. C.).

Shepherdson, J. C. Inverses and zero divisors in matrix rings. Proc. London Math. Soc. (3) 1, 71–85 (1951).

Let $A=(a_{ij})$, $B=(b_{ij})$ be two 2×2 matrices. Then the conditions that $AB=I$ are

$$\text{(C)}\qquad \begin{aligned} E_{11}&=a_{11}b_{11}+a_{12}b_{21}-1=0,\\ E_{12}&=a_{11}b_{12}+a_{12}b_{22}=0,\\ E_{21}&=a_{21}b_{11}+a_{22}b_{21}=0,\\ E_{22}&=a_{21}b_{12}+a_{22}b_{22}-1=0.\end{aligned}$$

The author considers the most general associative ring R with unit 1 satisfying conditions (C) and generated by the a_{ij} and b_{ij}. R is the residue class ring of the corresponding free ring modulo the two sided ideal generated by the E_{ij}. For free rings he gives an effective procedure for deciding whether an element $F(x_1, \cdots, x_n)$ belongs to the two-sided ideal generated by elements $E_i(x_1, \cdots, x_n)$, $i=1, \cdots, m$, providing the elements E_i are of a certain form. This procedure applies to the ideal defined above and shows that the ring R so defined has no zero divisors, and also that no one of the four equations required to make $BA=I$ is valid. Hence for this ring $X=BA-I\neq 0$, and we have $AB=I$, $AX=0$, $X\neq 0$. *Marshall Hall.*

Levitzki, Jakob. A note on prime ideals. Riveon Lematematika 5, 1–4 (1951). (Hebrew. English summary)

The author has proved in an earlier paper [Amer. J. Math. 73, 25–29 (1951); these Rev. 12, 474] that the intersection of all the prime ideals of an arbitrary ring R is the lower radical; the proof made use of some results of McCoy [ibid. 71, 823–833 (1949); these Rev. 11, 311]. In the present note a direct proof is given, making use of the following lemma: If A and C are ideals in R such that C is radical and $A\supset C$, then there exists a prime ideal P such that $P\supseteq C$, $P\not\supseteq A$. The lemma is proved by constructing two sequences $\{a_i\}$, $\{b_i\}$ of elements of R such that $a_i=a_{i-1}b_{i-1}a_{i-1}$, $i=1, 2, \cdots$, $a_0\epsilon A$, $a_iR\not\subseteq C$, $i=0, 1, 2, \cdots$. The ideal P is then taken as any ideal maximal with respect to the property of containing C but no a_i. *I. S. Cohen.*

Nakayama, Tadasi. Non-normal Galois theory for non-commutative and non-semisimple rings. Canadian J. Math. 3, 208–218 (1951).

Ce travail, écrit indépendamment d'un mémoire de G. Hochschild [Amer. J. Math. 71, 443–460 (1949); ces Rev. 10, 676] se développe suivant les mêmes idées. Dans une première partie, l'auteur étend le théorème de Jacobson-Bourbaki [Hochschild, loc. cit., th. 2.1] aux anneaux R satisfaisant à la condition minimale pour les idéaux à gauche. Dans la seconde partie, on définit le "module des relations" d'un bimodule cyclique sur un tel anneau R, en utilisant une base du bimodule considéré, mais la définition intrinsèque donnée par Hochschild [loc. cit., n° 5] s'applique en réalité à des bimodules cycliques sur un anneau quelconque. D'ailleurs, à l'exception du th. 5.3 de Hochschild [loc. cit.], les démonstrations de ses autres résultats ne font pas intervenir que les anneaux d'opérateurs sont des corps; il n'y intervient que le théorème de Jacobson-Bourbaki et éventuellement des hypothèses d'existence de bases; d'où les généralisations de ces théorèmes au cas considéré par l'auteur. *J. Dieudonné* (Nancy).

Bruck, R. H., and Kleinfeld, Erwin. The structure of alternative division rings. Proc. Nat. Acad. Sci. U. S. A. 37, 88–90 (1951).

An alternative division ring of characteristic $\neq 2$ is either 1) associative or 2) a Cayley-Dickson algebra over its (associative) center. This is an outline of the authors' proof of this important result, which will later appear in detail. The same result for characteristic $\neq 2, 3$ has been obtained independently and in a different way by L. A. Skornyakov [Ukrain. Mat. Žurnal 2, 70–85 (1950); these Rev. 12, 668]. Bruck and Kleinfeld prove somewhat more than the result above. They consider a ring R which (a) is alternative, (b) is not associative, (c) has no divisors of zero, and (d) has characteristic $\neq 2$. Such a ring R has a center C (elements both permuting and associating with all other elements) not identically 0. Every element of R satisfies a quadratic equation over C. If we extend R by the quotient field F of C, then R becomes a Cayley-Dickson algebra over F. *Marshall Hall* (Washington, D. C.).

Kaplansky, Irving. Semi-simple alternative rings. Portugaliae Math. 10, 37–50 (1951).

This paper, following upon the lead of Bruck and Kleinfeld [see review above], investigates alternative rings without assumption of chain conditions. But for the most part one of two qualitative assumptions is made: 1) a hypothesis due to Zorn: Every element x is either nilpotent or has a right multiple xy which is a nonzero idempotent; or 2) π-regularity: For every element x, there exists an integer n and an element y such that $x^nyx^n=x^n$. The second of these hypotheses includes and is stronger than the first. These assumptions assure a supply of idempotents by means of which the structure of the rings can be attacked. A study of the Peirce decomposition (following A. A. Albert) is made in some detail. The radical (M. F. Smiley's extension to alternative rings of N. Jacobson's definition of radical) is contained in every regular maximal right ideal, and with the Zorn hypothesis the radical is the intersection of the regular maximal right ideals. For π-regular rings and M a maximal right ideal either 1) M is two-sided and A/M is a division ring or a Cayley matrix ring (8 units) over a field or 2) if P is the set of all x with $Ax\subset M$, then $P\subset M$, P is an ideal, and A/P is a primitive associative ring. Finally it is shown that a compact semi-simple alternative ring is a complete direct sum of finite simple rings. *Marshall Hall.*

Price, Charles M. Jordan division algebras and the algebras $A(\lambda)$. Trans. Amer. Math. Soc. 70, 291–300 (1951).

The author studies central Jordan division algebras of finite rank over a field of characteristic 0. Since every such algebra is simple, the known structure of the central simple Jordan algebras can be used and it remains to investigate which of these simple algebras are actually division algebras. A number of results in this direction are obtained. We give here some of these in the case of a central simple Jordan algebra B of the first type. Such an algebra B consists of the same elements as a central simple associative algebra A, but if the product in A is denoted by xy, the product in B is given by $x\circ y=(xy+yx)/2$. If B is to be a division algebra, then A must be a division algebra. If A is a central division algebra of odd rank over its center, then B is a division algebra. On the other hand, if the rank is even and if it is known that A can be written as a crossed product, then B is not a division algebra. (It is still unknown whether every central simple associative algebra can be written as a crossed product.) The author also considers nonassociative algebras obtained from A by defining a new multiplication by $x\circ y=\lambda xy+(1-\lambda)yx$ where λ is a fixed element of the underlying field. While the results of the author do not settle the whole question completely, they suffice at least in the case that the ground field is an algebraic number field of finite degree. *R. Brauer* (Ann Arbor, Mich.).

Kähler, Erich. Über rein algebraische Körper. Math. Nachr. 5, 69–92 (1951).

The author states at the end of the introduction: "Die vorliegende Arbeit versucht, zur Erforschung der rein algebraischen Körper beizutragen." Unfortunately the statements of the author on algebraic function fields which are

finitely generated over an arbitrary prime field are not backed up in this paper by proofs or detailed references to proofs of some of the known theorems concerning differential forms, which are contained in the literature.
O. F. G. Schilling (Chicago, Ill.).

Moyls, B. N. The structure of valuations of the rational function field $K(x)$. Trans. Amer. Math. Soc. 71, 102–112 (1951).

Suppose that the field K admits the valuation V with the value group VK and the residue class field $\underline{K}$. The author investigates the possible structures of the prolongations of V to a purely transcendental extension $K(x)$, and proves several existence theorems concerning prolongations with prescribed structure. The restriction to purely transcendental extensions places strong limitations on the transcendence degree $T(\underline{L}/\underline{K})$, where L denotes the residue class field of the particular prolongation, $T(\underline{L}/\underline{K})+R(\Gamma/VK)\leqq 1$, where $R(.\,.)$ denotes the rational rank of the factor group of the value group of the prolongation Γ modulo VK. Detailed results which are arrived at by extended use of MacLane's theory of augmented valuations and key polynomials [see especially S. MacLane, Trans. Amer. Math. Soc. 40, 363–395 (1936)], based on Gauss' Lemma, are proved under the additional finiteness and denumerability assumptions. This approach which does not require the consideration of the algebraic completion of K, as does the theory of pseudo-convergent sequences of Ostrowsky, is "elementary" in the sense that it exploits over and over again the process of division with remainder for polynomials combined with some transfinite constructions which are carried out with due care. There are no restrictions on the valuation V.
O. F. G. Schilling (Chicago, Ill.).

Theory of Groups

Bruck, R. H. On a theorem of R. Moufang. Proc. Amer. Math. Soc. 2, 144–145 (1951).

Ref. hat in einer früheren Arbeit [Math. Ann. 110, 416–430 (1935)] bewiesen, dass in einem loop, in dem die Identität $xy\cdot zx=(x\cdot yz)x$ gilt, drei Elemente a, b, c eine Gruppe erzeugen, falls sie der Relation $(ab)c=a(bc)$ genügen. Für den Specialfall eines kommutativen loop gibt Verf. einen einfachen Beweis dieses Satzes. *R. Moufang.*

***Piccard, Sophie. Sur les groupes d'ordre fini.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 211–215. Centre National de la Recherche Scientifique, Paris, 1950.

This paper summarizes succinctly the researches of the author on the minimal bases of systems of generators of a finite group G. An indication is given of the significance of such a basis for the subgroup structure of G and also of the relationship between the different possible bases of G. The problem of the bases of the symmetric and alternating groups is considered in some detail, so far as it has been solved up to the present time. *G. de B. Robinson.*

Amante, S. I gruppi finiti dei tipi 5 e 6. Matematiche, Catania 4, 1–20 (1949).

This is a continuation of a previous paper of the author [Rend. Accad. Sci. Fis. Mat. Napoli (3) 36, 155–165 (1930)], completing the characterisation of all finite groups in which the normalizer of each element is abelian. These groups are exactly those of "type" $\leqq 6$, where the type means the number diminished by 2 of the distinct normalizers of all single elements of the group. Whereas in the cited paper the author gives a full characterisation of the groups of type $\leqq 4$, in this paper he deals with the cases 5, 6. The author succeeds in the more detailed investigation of the groups of the type 5 and 6 by proving first that the index of the centrum of such a group can take on the values 8, 12, 16, resp. 14, 21, 42, 49 only. *T. Szele* (Debrecen).

Amante, S. I sottogruppi fondamentali di gruppi quasi-abeliani. Matematiche, Catania 4, 21–36 (1949).

Non-abelian p-groups with only commutable pairs of subgroups and having a cyclic invariant subgroup with cyclic quotient group are investigated. In particular, the author determines the normalizer of each element of such a group and the number of all distinct normalizers among these. *T. Szele* (Debrecen).

***Conrad, Paul F. Imbedding theorems for Abelian groups with valuations.** Abstract of a Thesis, University of Illinois, 1951. i+3 pp.

***Hirsch, K. A. Sur les groupes résolubles à condition maximale.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 209–210. Centre National de la Recherche Scientifique, Paris, 1950.

Takahasi, Mutuo. Note on locally free groups. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. 1, 65–70 (1950).

A group G is said to be locally free if and only if every finite set of elements of G generates a free group. The number of free generators of a free group F is called the rank of F. A subgroup K of a locally free group G is called a *-subgroup of G if and only if K is a free factor of every free group F of finite rank which satisfies $G\supseteq F\supseteq K$. The author proves that each of the following two conditions is necessary and sufficient for a countable locally free group G to be a free group: 1) Every finite set of elements of G can be embedded into a *-subgroup of G. 2) If $V_1\subset\cdots\subset V_n\subset\cdots$ is an infinite properly increasing sequence of free subgroups of finite rank of G, then V_1 must be contained in a proper free factor of some V_n. *F. W. Levi* (Bombay).

***Tits, J. Groupes triplement transitifs et généralisations.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 207–208. Centre National de la Recherche Scientifique, Paris, 1950.

A group of transformations is n-tuply transitive if there exists a unique element of the group that transforms n arbitrary pairwise distinct points into any prescribed n-tuple of points. Such groups have been studied by the author for $n=3$ in a series of four papers [Acad. Roy. Belgique. Bull. Cl. Sci. (5) 35, 197–208, 224–233, 568–589, 756–773 (1949); these Rev. 11, 9, 320]. For $n>3$ the only n-tuply transitive groups are (1) symmetric groups of degree $k\geqq 4$, (2) alternating groups of degree $k\geqq 6$, (3) the Mathieu group of degree 12, and (4) the quadruply transitive group obtained from it by fixing one of the points. In this note the notion of "almost n-tuply transitive group" is defined and some elementary properties of such groups are stated. A group of transformations is almost n-tuply transitive provided (1)

there corresponds to each $n-1$ points at least one transformation of the group (different from the identity) that preserves them individually, while n-tuples (called nonsingular) exist without this property, and (2) there exists a unique transformation of the group that carries an arbitrary nonsingular n-tuple into any prescribed nonsingular n-tuple. (The collineations of the plane, for example, are merely almost quadruply transitive.) *L. M. Blumenthal.*

Dynkin, E. B. The relations of inclusion between irreducible groups of linear transformations. Doklady Akad. Nauk SSSR (N.S.) **78**, 5–7 (1951). (Russian)

The determination of all inclusion relations between connected irreducible groups of unimodular complex linear transformations is announced. The problem is reduced to the case in which the containing group is simple by showing that every group G of the type under consideration is the tensor product $G_1 \times G_2 \times \cdots \times G_r$ of groups of the same type that are simple; and that moreover, a subgroup H of G is of this type if and only if $H = H_1 \times H_2 \times \cdots \times H_r$ with the H_i of the same type and $H_i \subset G_i$. A table enumerates all subgroups of the type specified, within the usual kind of equivalence, for all simple groups (of the same type), excluding the trivial case $SL(n)$ and the cases $O(n)$ and $Sp(n)$, which have been treated by Malcev [Izvestiya Akad. Nauk SSSR. Ser. Mat. **8**, 143–174 (1944); Amer. Math. Soc. Translation no. **33** (1950); these Rev. **6**, 146; **12**, 317]. The author states that the present results permit the enumeration of the maximal subgroups of the classical groups and have other applications. *I. E. Segal* (Princeton, N. J.).

Putnam, Calvin R., and Wintner, Aurel. The connectedness of the orthogonal group in Hilbert space. Proc. Nat. Acad. Sci. U. S. A. **37**, 110–112 (1951).

The orthogonal group on a finite-dimensional real Hilbert space is not connected, as there is no arc joining a reflection and the identity. It is shown that when the space is infinite-dimensional, there is always such an arc, and from this fact it is deduced that the group is in this case arcwise connected. *I. E. Segal* (Princeton, N. J.).

Chowla, S., Herstein, I. N., and Moore, W. K. On recursions connected with symmetric groups. I. Canadian J. Math. **3**, 328–334 (1951).

Denote by T_n the number of solutions of $x^2=1$ in φ_n, the symmetric group of degree n. The authors investigate the asymptotic properties of T_n. They first of all show that (1) $T_n = T_{n-1} + (n-1)T_{n-2}$. From (1) they deduce that $n^{\frac{1}{2}} < T_n/T_{n-1} < n^{\frac{1}{2}}+1$, and finally prove that

$$T_n = (1+o(1))\frac{(n/e)^{n/2}e^{n^{\frac{1}{2}}}}{2^{\frac{1}{2}}e^{\frac{1}{4}}}.$$

P. Erdös (Aberdeen).

Newell, M. J. On the quotients of alternants and the symmetric group. Proc. London Math. Soc. (2) **53**, 345–355 (1951).

The author reviews the familiar expressions for the characters of the irreducible representations of the symmetric group using the quotient matrix of two alternant matrices. *G. de B. Robinson* (Toronto, Ont.).

Steinberg, Robert. The representations of GL(3, q), GL(4, q), PGL(3, q), and PGL(4, q). Canadian J. Math. **3**, 225–235 (1951).

Let GL(n, q) denote the group of all regular substitutions with marks in the Galois field GF(q), and PGL(n, q) the corresponding projective group, i.e. PGL(n, q)$\cong$GL(n, q)/H, where H is the group of the $q-1$ scalar matrices. The author determines the irreducible characters of GL(n, q) and PGL(n, q) for $2 \leqq n \leqq 4$. Essential use is made of the properties of the geometry PG($n-1, q$) of which PGL(n, q) is the collineation group. Each element of GL(n, q) may be regarded as a linear transformation in PG($n-1, q$), and this transformation induces a permutation of the entity of points in PG($n-1, q$). The permutation representations thus obtained are for $n=2, 3, 4$ of degrees $q+1$, q^2+q+1, q^3+q^2+q+1 respectively, and they split into a unit representation and an irreducible representation. Multiplication of the irreducible characters thus obtained by the linear characters defined by the powers of the determinants, yields additional irreducible characters. For $n \geqq 3$ this procedure is repeated for the totality of all configurations of a line through a point, and finally for $n=4$ also for the totality of all configurations consisting of a point, a line through it, and a plane through the line. To obtain the remaining irreducible characters of GL(n, q) the author repeatedly applies the well known device of the so called "induced characters", by considering the linear characters of certain subgroups which are then extended to characters of the whole group. As to the characters of PGL(n, q), these are obtained from those of GL(n, q) with the help of the simple relation PGL(n, q)$\cong$GL(n, q)/H mentioned above. *J. Levitzki* (Jerusalem).

Itô, Noboru. Some studies on group characters. Nagoya Math. J. **2**, 17–28 (1951).

Let G be a group of finite order g which has a normal subgroup N of prime index. A construction is given by which the blocks of representations of G are associated with blocks of N [cf. R. Brauer, Proc. Nat. Acad. Sci. U. S. A. **30**, 109–114 (1944); **32**, 182–186, 215–219 (1946); these Rev. **6**, 34; **8**, 14, 131]. As an application, the author obtains the following results for soluble groups G of finite order g: If G has no normal subgroup whose order is a power $p^r > 1$ of a fixed prime p which divides g, then there exists a p-block whose defect is not the maximal one. The same conclusion holds, if the p-Sylow group of G is abelian and not normal in G. If all p-Sylow groups in G have the intersection 1, then there exists a p-block of defect 0. If the degrees of all irreducible representations are prime to p, then the p-Sylow group in G is normal. If $g = p^a g'$ with $(g', p) = 1$ and if $g \leqq p^{2a}$, then G has no p-block of defect 0. [The last result holds even if G is not soluble. It is a trivial consequence of the relation which states that g equals the sum of the squares of the degrees of the irreducible representations.] *R. Brauer* (Ann Arbor, Mich.).

Johnston, Francis E. The theory of group representations. J. Washington Acad. Sci. **41**, 117–129 (1951).

This is a descriptive address concerning the elements of group representation theory. Starting from the definition of a group it goes on to discuss matrix representations of finite groups, permutation groups, symmetric groups, rotation groups, unitary groups. Some space is devoted to spin representations of orthogonal groups. It ends with a brief indication of the importance of the theory in relation to Schrödinger's equation and quantum mechanics. *D. E. Littlewood* (Bangor).

Yoshizawa, Hisaaki. Some remarks on unitary representations of the free group. Osaka Math. J. **3**, 55–63 (1951).

Soit G le groupe libre à deux générateurs, lesquels sont notés a et b dans ce qui suit. L'auteur montre d'abord qu'il

est impossible d'approcher la fonction 1 par des fonctions de la forme $f*\tilde{f}$ (f de carré sommable sur G), ce qui résout négativement un problème posé par le rapporteur [Trans. Amer. Math. Soc. 63, 1–84 (1948); ces Rev. 9, 327]. Ceci fait, soit P_1 l'ensemble des fonctions de type positif sur G telles que $f(e)=1$, ensemble muni de la topologie faible usuelle; l'auteur prouve alors l'existence d'une représentation unitaire irréductible $x\rightarrow U_x$ de G telle que les fonctions élémentaires de type positif de la forme $(U_x\mathbf{a}, \mathbf{a})$ ($\mathbf{a}$ vecteur unité arbitraire dans l'espace de cette représentation) soient partout denses dans P_1; cet exemple montre que la généralisation du théorème de Bochner donnée par le rapporteur dans l'article cité peut être triviale (mais aussi montre que la généralisation donnée plus récemment par Mautner et par le rapporteur ne l'est pas!). Enfin, soit $x\rightarrow U_x$ la représentation régulière de G par les translations à droite dans $L^2(G)$; soit A (resp. B) le sous-groupe abélien des a^n (resp. b^n); si l'on décompose la représentation en question par une transformation de Fourier sur A (resp. sur B), on obtient deux décompositions de celle-ci en somme continue de représentations irréductibles, avec la propriété suivante, dont le caractère pathologique est malheureusement éclatant: toute composante de la première décomposition est distincte (au sens de l'équivalence unitaire) de toute composante de la seconde. Cela confirme évidemment un fait plus ou moins connu déjà, à savoir que, du point de vue de la théorie des représentations unitaires, les groupes généraux ont un comportement pathologique; il faut toutefois remarquer que ces phénomènes ne se présentent pas quand on veut étendre à ces groupes la théorie des caractères. *R. Godement.*

Mautner, F. I. The regular representation of a restricted direct product of finite groups. Trans. Amer. Math. Soc. 70, 531–548 (1951).

Soit G un groupe discret dénombrable; dans $L^2(G)$, considérons l'anneau faiblement fermé $\mathbf{R}^s$ engendré par les translations à gauche U_s: $f(x)\rightarrow f(s^{-1}x)$, et soit $\mathbf{Z}$ le centre de cet anneau; soit $\int\oplus H(t)$ la décomposition de $L^2(G)$ relativement à $\mathbf{Z}$, au sens de von Neumann; alors $\mathbf{R}^s$ se décompose en anneaux $\mathbf{R}^s(t)$ qui, G étant discret, sont presque tous des facteurs de classe finie (I_n, n fini, ou II_1); l'auteur pose le problème suivant: à quelle condition ces facteurs sont-ils presque tous de classe II_1? et le résoud pour une classe particulière de groupes, à savoir quand G est le produit direct d'une suite G_n de groupes finis (les éléments de G sont les suites (x_n) dont "presque tous" les éléments sont égaux à e). La réponse est qu'une infinité de groupes G_n doivent être non commutatifs.

L'auteur montre aussi que, dans ce cas, la décomposition centrale de la représentation régulière de G fait intervenir tous les "caractères" de G (un caractère de G étant une fonction centrale qui définit un homomorphisme dans le corps complexe de l'algèbre des fonctions centrales sommables sur G) et remarque que cette assertion n'a de sens que parce qu'il a pu introduire dans l'ensemble de ces caractères une topologie naturelle (il est en effet évident que, si l'on utilisait des décompositions purement "measure theoretic", la question n'aurait aucun sens); il est facile de voir que la même propriété a lieu pour tous les "groupes centraux" du rapporteur, lesquels contiennent les groupes étudiés par l'auteur. Quant au problème de déterminer quels caractères interviennent dans la décomposition de la représentation régulière d'un groupe G, on peut lui donner un sens et le résoudre pour tous les groupes discrets dénombrables (réponse: un caractère de G (discret) intervient dans la représentation régulière si et seulement si on peut l'approcher par des fonctions centrales, de type positif et de carré sommable); ce résultat, et des résultats analogues valables pour tout groupe séparable unimodulaire, seront publiés prochainement par le rapporteur; bien entendu, les démonstrations de théorèmes de ce genre exigent absolument l'introduction de topologies naturelles dans les ensembles de caractères considérés, comme il est facile de s'en convaincre dans les cas les plus classiques. *R. Godement* (Nancy).

Mautner, F. I. On the decomposition of unitary representations of Lie groups. Proc. Amer. Math. Soc. 2, 490–496 (1951).

Soient G un groupe de Lie connexe et $x\rightarrow U_x$ une représentation unitaire de G dans un espace de Hilbert $\mathfrak{H}$; à toute transformation infinitésimale X sur G correspond alors dans $\mathfrak{H}$ un opérateur self-adjoint (en général non borné) $\bar{X}$ tel que $x=\exp(tX)$ implique $U_x=\exp(it\bar{X})$; de plus, on sait que ces $\bar{X}$ sont tous définis sur une sous-espace partout dense de $\mathfrak{H}$. Maintenant, supposons donnée une décomposition $\mathfrak{H}=\int\oplus\mathfrak{H}(t)$ de $\mathfrak{H}$ (au sens de von Neumann) pour laquelle chaque U_x se décompose; l'auteur montre que les "champs d'opérateurs" $U_x(t)$ associés aux U_x (et qui ne sont définis à priori qu'à des ensembles de mesure nulle près) peuvent alors être choisis de telle sorte que, pour presque chaque t, $x\rightarrow U_x(t)$ soit une représentation unitaire continue de G; de plus, chaque opérateur $\bar{X}$ se décompose aussi en opérateurs $\bar{X}(t)$, et on peut les choisir de telle sorte que, pour presque chaque t, ces $\bar{X}(t)$ aient un domaine de définition commun $\mathfrak{D}(t)$ partout dense dans $\mathfrak{H}(t)$ avec la condition que $X\rightarrow\bar{X}(t)$ soit, pour ces t non exceptionnels, une représentation linéaire dans $\mathfrak{D}(t)$ de l'algèbre de Lie de G.

La méthode de l'auteur consiste à démontrer d'abord l'assertion relative aux $\bar{X}(t)$, puis à en déduire celle qui se rapporte aux $U_x(t)$; or cette dernière propriété n'a aucun rapport avec la théorie des groupes de Lie, comme nous allons le montrer. Soit dx la mesure de Haar à gauche de G, et L^1 l'algèbre des fonctions sommables pour dx; pour $f\in L^1$ posons $U_f=\int U_x\cdot f(x)dx$; les U_x étant décomposables, il en est évidemment de même des U_f. Le groupe G étant supposé séparable, il existe dans L^1 une suite partout dense (f_n), et on peut évidemment supposer que les f_n forment une algèbre sur le corps K des nombres complexes rationnels, stable par l'involution $f\rightarrow\tilde{f}$ de L^1. Ceci dit, choisissons une fois pour toutes (de façon quelconque) les champs d'opérateurs $U_{f_n}(t)$ associés aux U_{f_n}; comme toute relation algébrique entre les f_n se traduit presque partout sur les $U_{f_n}(t)$, et comme celles de ces relations qui sont à coefficients dans K sont en infinité dénombrable, il existe un ensemble N de mesure nulle tel que, pour t non-$\in N$, l'application $f_n\rightarrow U_{f_n}(t)$ soit compatible avec toutes les relations algébriques à coefficients dans K entre les f_n, et avec l'involution $\tilde{}$; de plus, pour chaque n, on a $\|U_{f_n}(t)\|\leqq\|U_{f_n}\|\leqq\|f_n\|_1$ presque partout, de sorte qu'on peut supposer N choisi de telle sorte que t non-$\in N$ implique $\|U_{f_n}(t)\|\leqq\|f_n\|_1$ pour tout n. Il résulte de là que, pour t non-$\in N$, $f_n\rightarrow U_{f_n}(t)$ se prolonge par continuité en une application $f\rightarrow U_f(t)$ de L^1, laquelle est continue, compatible avec les relations algébriques à coefficients dans K (donc aussi avec les relations à coefficients complexes), et compatible avec l'involution de L^1; autrement dit, $f\rightarrow U_f(t)$ est, pour t non-$\in N$, une représentation unitaire de L^1; et évidemment, l'opérateur U_f de $\mathfrak{H}$ se décompose suivant les $U_f(t)$.

Posons maintenant $f_s(x)=f(s^{-1}x)$; il est clair qu'on a $U_sU_f=U_{f_s}$; réciproquement, soit $f\rightarrow U_f$ une représentation unitaire de L^1 dans un espace $\mathfrak{H}$, et supposons-la non

dégénérée (i.e. les vecteurs $U_f\mathfrak{x}$, $f\epsilon L^1$, $\mathfrak{x}\epsilon\mathfrak{H}$, engendrent tout $\mathfrak{H}$); alors il est visible qu'on peut définir une représentation de G par la formule $U_s(U_f\mathfrak{x})=U_{f_s}\mathfrak{x}$, et que $U_f=\int U_s f(x)dx$. Revenant au problème qui nous occupe, on voit facilement que, la représentation $f\rightarrow U_f$ étant non dégénérée, il en est de même de $f\rightarrow U_f(t)$ pour presque tout t; donc on peut supposer N choisi de telle sorte que, pour t non-ϵN, la représentation $f\rightarrow U_f(t)$ soit engendrée par une représentation unitaire continue $x\rightarrow U_x(t)$ de G, et il reste à vérifier que U_x se décompose suivant les $U_x(t)$, ce qui est trivial.

L'auteur dit que son théorème implique, pour G connexe de Lie, la possibilité de représenter toute fonction continue de type positif sur G par une intégrale portant sur des fonctions élémentaires continues de type positif; il est clair que, d'après ce qui précède, cette possibilité a lieu pour tout groupe séparable (de Lie ou non), et cela résulte aussi trivialement d'un article récent du rapporteur [Ann. of Math. (2) 53, 68–124 (1951); ces Rev. 12, 421; voir en particulier la formule (203), p. 112]. *R. Godement.*

Godement, Roger. Mémoire sur la théorie des caractères dans les groupes localement compacts unimodulaires. J. Math. Pures Appl. (9) 30, 1–110 (1951).

This is primarily a detailed exposition of results previously announced [C. R. Acad. Sci. Paris **229**, 967–969, 1050–1051, 1107–1109 (1949); these Rev. 11, 325]. It contains also a discussion of a type of trace on the enveloping algebra of a semi-simple Lie group, an exposition of the background and problems of the field, and some comments of recent date concerning the author's investigations in the light of more recent work. The author states in particular that Mackey has given examples of groups having characters in a certain natural sense which are not characters in the sense of the present paper. *I. E. Segal* (Princeton, N. J.).

Hochschild, G. Group extensions of Lie groups. Ann. of Math. (2) 54, 96–109 (1951).

Let G, H be connected Lie groups and (E, π) an extension of G by H, consisting of a connected Lie group E and a homomorphism π of E onto H, whose kernel coincides with G. If G is abelian, such an extension uniquely determines a homomorphism π^0 of H into the group $A(G)$ of all automorphisms of G. Now let η be a fixed homomorphism of H into $A(G)$. Defining the equivalence of two extensions suitably, the author proves, without using factor sets, that those equivalent classes of extensions (E, π) of G by H, which give the same $\pi^0=\eta$, make up an abelian group Ext (G, H, η). (When η is the trivial homomorphism this group is simply denoted by Ext (G, H).) The author then studies the structure of the group Ext (G, H, η) under various conditions on H. In particular, he proves that, if H is semi-simple, Ext (G, H) is isomorphic to the group of homomorphisms of the fundamental group of H into G, and if, moreover, this fundamental group is finite, then Ext (G, H, η) is isomorphic to Ext (T, H), where T is the maximal compact subgroup of G.

For the proof, the author introduces lifted extensions defined as follows: Let G, H and (E, π) be as above, and let L be a connected Lie group and ω a homomorphism of L onto H with the kernel K. Then, there exists an extension (U, λ) of G by L with the following properties: U is mapped by a homomorphism ϵ onto E such that $\epsilon(g)=g$ for all $g\epsilon G$ and that $\pi\epsilon=\omega\lambda$. He then proves that the mapping $(E, \pi)\rightarrow(U, \lambda)$ induces a homomorphism of Ext (G, H, η) into Ext $(G, L, \eta\omega)$ and that the kernel of this homomorphism can be expressed in terms of K, L and G. Applying this formula to a semi-simple group H and its simply connected covering group L, and using some properties on simply connected Lie groups, he proves the above mentioned theorems on Ext (G, H, η). The author, also, notes that the usual method of factor sets is, in general, not suitable for extensions of Lie groups, and gives a simple example of such an extension, which has no continuous factor set. *K. Iwasawa* (Princeton, N. J.).

de Siebenthal, Jean. Sur les sous-groupes de rang un des groupes de Lie clos. C. R. Acad. Sci. Paris **232**, 1892–1893 (1951).

Two theorems are stated by means of which it is possible to determine the simple Lie subgroups of rank 1 of a compact connected Lie group G. It is sufficient to determine those subgroups g of rank 1 which are not contained in proper subgroups of rank l (=rank of G); for such a g the author writes $g\underset{II}{\subset}G$. A subgroup g is called singular or regular according as it contains ~~singular~~ regular elements of G or not. The first theorem asserts that every non-commutative G contains a regular $g\underset{II}{\subset}G$. The second theorem makes precise the proposition that the singular g's, $g\underset{II}{\subset}G$, only occur in exceptional cases. *P. A. Smith* (New York, N. Y.).

Murakami, Shingo. Remarks on the structure of maximally almost periodic groups. Osaka Math. J. 2, 119–129 (1950).

L'auteur démontre le théorème suivant: soit G un groupe localement compact, G_0 la composante connexe de l'unité dans G, et supposons G/G_0 compact; alors G est maximalement presque périodique si et seulement si G contient un sous-groupe compact K et un sous-groupe invariant V isomorphe à un R^n tels que $G=KV$, $K\cap V=e$, tout élément de V devant en outre commuter à tout élément de la composante connexe de e dans K. Ce théorème généralise celui de Freudenthal sur les groupes connexes, et sa démonstration utilise d'ailleurs ce cas particulier. L'auteur donne ensuite un exemple d'un groupe maximalement presque périodique qui ne possède pas de structure uniforme bi-invariante; ce groupe est totalement discontinu et ne possède aucun sous-groupe ouvert compact. *R. Godement.*

Kuranishi, Masatake. On non-connected maximally almost periodic groups. Tôhoku Math. J. (2) 2, 40–46 (1950).

Soient G un groupe localement compact et G_0 la composante connexe de l'unité dans G; supposons G/G_0 compact. Alors, si G_0 est maximalement presque périodique, G contient un groupe compact K et un groupe V isomorphe à un R^n tels que $G=KV$, $K\cap V=e$. De plus, l'hypothèse faite sur G_0 équivaut au fait que G est lui-même maximalement presque périodique. Il y a lieu de rapprocher ces résultats de ceux de Murakami [voir l'analyse précédente], qui sont plus précis. *R. Godement* (Nancy).

Chevalley, C. On a theorem of Gleason. Proc. Amer. Math. Soc. 2, 122–125 (1951).

L'auteur démontre le théorème suivant: soit G un groupe localement euclidien ne contenant pas de sous-groupes "arbitrairement petits"; alors il existe un voisinage P de e dans G tel que, pour tout $t\epsilon P$, il existe un et un seul sous-groupe à un paramètre $x\rightarrow s_t(x)$ vérifiant $s_t(x)\epsilon P$ pour $-1\leqq x\leqq 1$, $s_t(1)=t$; en outre $s_t(x)$ est une fonction continue de (t, x). La démonstration procède comme suit. Tout d'abord, un résultat de Gleason montre l'existence d'un

voisinage symétrique compact V de e ne contenant aucun sous-groupe $\neq e$ de G et sur lequel l'application $s\rightarrow s^2$ est biunivoque; pour tout entier $n>0$, soit Q_n l'ensemble des $u\varepsilon V$ avec $u^m\varepsilon V$ pour $|m|\leqq 2^n$, et soit P_n ($\subset V$) l'image de Q_n par $s\rightarrow s^{2^n}$; on note f_n l'application de P_n sur Q_n réciproque de la précédente, et on pose $P=\bigcap P_n$. Soit alors $t\varepsilon P$; soit $x\varepsilon[-1,+1]$; soit $\mathfrak{F}$ un ultrafiltre non trivial sur l'ensemble des entiers, et choisissons des entiers m_n tels que $|m_n|\leqq 2^n$, $\lim_{\mathfrak{F}} 2^{-n}m_n=x$; alors $s_t(x)=\lim_{\mathfrak{F}}(f_n(t))^{m_n}$ existe par compacité; l'auteur montre que cette limite ne dépend pas de la suite (m_n), et définit de façon évidente un sous-groupe à un paramètre de G. La continuité de l'application $(x,t)\rightarrow s_t(x)$ se démontre facilement, ainsi que l'unicité du sous-groupe passant par t. Reste à montrer que P est un voisinage de e, ce que utilise essentiellement le théorème de l'invariance du domaine. *R. Godement* (Nancy).

Yamabe, Hidehiko. Note on locally compact groups. Osaka Math. J. 3, 77–82 (1951).

This paper contains the following two theorems. Theorem 1. A locally Euclidean group G, which has a neighbourhood of the identity containing no non-trivial subgroup, has a neighbourhood U^* of the identity through any point of which there exists one and only one one-parameter subgroup. Theorem 2. If $(U_n)^n$ is contained in U^*, then G is a Lie group, where U_n denotes the aggregate of the nth roots of elements in a neighbourhood U. The author remarks that theorem 1 was proved with the cooperation of Gotô. This theorem has been proved independently by Chevalley [see the preceding review], and by Gleason whose proof is unpublished.
D. Montgomery (Princeton, N. J.).

Gotô, Morikuni, and Yamabe, Hidehiko. On some properties of locally compact groups with no small subgroup. Nagoya Math. J. 2, 29–33 (1951).

This paper shows that in a locally Euclidean group with no small subgroup every point sufficiently near the identity is on a unique one-parameter subgroup. This is theorem 1 mentioned in the review above where additional references are given. Theorem 2. Let G be a locally compact group with no small subgroup, and let L be a closed invariant subgroup of G. If L is a Lie group, then the factor group G/L has no small subgroup. Theorem 3. Let G be a locally compact group with no small subgroup, and H a closed subgroup. If H is a maximal connected Lie group in G, then the identity component of the normalizer $n(H)$ coincides with H. Theorem 4. Let G be a locally compact group with no small subgroup, and H_1 a closed local subgroup. If H_1 is a maximal local Lie group, then the identity component of the normalizer of H_1 coincides with H_1 locally. *D. Montgomery.*

Gotô, Morikuni. On local Lie groups in a locally compact group. Ann. of Math. (2) 54, 94–95 (1951).

The theorem proved is the following. Let H be a locally compact group and L_1 a closed local Lie group in H. Let L be the group generated by L_1 in H. Then the closure of L is an (L)-group in the sense of K. Iwasawa [Ann. of Math. (2) 50, 507–558 (1949); these Rev. 10, 679]. The theorem remains valid if the local Lie group L_1 is replaced by a local (L)-group.
P. A. Smith (New York, N. Y.).

Loonstra, F. Discrete groups. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 162–168 (1951).

Let G be a partially ordered group. A positive element of G is called minimal if it covers the unity of G; the set of all positive minimal elements is the covering set $D(0)$ of the unity. The group G is said to be discrete if the set of the elements between any element $a>0$ of G and the unity has minimal elements. The paper deals with the determination of lattice-ordered groups by properties of the covering sets $D(0)$ and other conditions. The main result is given by the theorem that a lattice-ordered group G which is discrete and archimedean is isomorphic with a subgroup of the cardinal product $\prod I$ of the factors I, I being the additive group of the integers. *O. Borůvka* (Brno).

NUMBER THEORY

★Chintschin, A. J. Drei Perlen der Zahlentheorie. Akademie-Verlag, Berlin, 1951. 62 pp. 6.50 DM.

A translation of A. Ya. Hinčin, Tri žemčužiny teorii čisel, 2d ed., OGIZ, Moscow-Leningrad, 1948; cf. these Rev. 11, 83.

McClenon, R. B. Bernoulli numbers. Proc. Iowa Acad. Sci. 57, 315–319 (1950).

Lehmer, D. H. A triangular number formula for the partition function. Scripta Math. 17, 17–19 (1951).

Using the elementary theory of formal power series together with Jacobi's formula for $\prod_{n=1}^{\infty}(1-x^n)^3$, the author proves that if $\prod_{n=1}^{\infty}(1-x^n)^\alpha=\sum_{m=0}^{\infty}C_\alpha(m)x^m$, then

$$\sum_{m=0}^{\infty}(-1)^m(2m+1)\times[n-(3+\alpha)(m+1)m/6]C_\alpha(n-\tfrac{1}{2}(m+1)m)=0.$$

The case $\alpha=-1$ yields an apparently new recurrence for the partition function, and $\alpha=24$ recovers one for Ramanujan's $\tau(n)$ [S. Ramanujan, Trans. Cambridge Philos. Soc. 22, 159–184 (1916)]. *N. J. Fine* (Philadelphia, Pa.).

Ward, Morgan. A class of soluble Diophantine equations. Proc. Nat. Acad. Sci. U. S. A. 37, 113–114 (1951).

Let $F(x_1,\cdots,x_l)$ be a homogeneous polynomial of degree n with integer coefficients, and let $m>0$ be prime to n. Then the diophantine equation $F(x)=z^m$ has the trivial family of integer solutions $x_i=y_iF^k(y)$, $z=F^l(y)$, where $k>0$, $l>0$ is any solution of $nk+1=lm$ and $y_1,\cdots,y_l$ are any integers.
J. W. S. Cassels (Cambridge, England).

Selmer, Ernst S. The Diophantine equation

$$ax^3+by^3+cz^3=0.$$

Acta Math. 85, 203–362 (1 plate) (1951).

The principal equations discussed are

$$(1)\qquad f(x,y,z)=ax^3+by^3+cz^3=0$$

where a, b, c are cube-free rational integers, relatively prime in pairs, (2) $x^3+y^3=Az^3$ with $A=abc$, and (3) $x^3-my^3=nz^3$ which is obtained from (1) by multiplying by a^2 and replacing ax by $-x$. Various known results are reviewed at the start, such as the connection between (1) and (2). It is indicated that recent work of Cassels [Acta Math. 82, 243–273 (1950); these Rev. 12, 11] can be used to demonstrate the impossibility (apart from $x=y=z=0$) of (1) in

certain cases. Further cases in which (1) is not solvable can be decided by simple congruence conditions, although Skolem [Avh. Norske Vid. Akad. Oslo. I. **1942**, no. 4; these Rev. **8**, 7] has proved that $f(x, y, z) \equiv 0 \pmod{p}$ is solvable for any prime $p > 7$ for which $(p, abc) = 1$.

The author gets at equation (1) largely through (3), which, factored in $R(\sqrt[3]{m})$, leads to equations of the form (4) $[x - y\sqrt[3]{m}] = \mathfrak{t}\mathfrak{a}^3$ where $\mathfrak{t}$ is an ideal from a finite set. The impossibility of (4) is established in various cases by class number considerations. If such an exclusion fails, then an auxiliary cubic, arising from comparison of coefficients of $m^{\frac{1}{3}}$ in (4), is studied. To facilitate discussion of this auxiliary cubic, an extensive examination is made of cubic residues in the field $R(\sqrt[3]{m})$, yielding further necessary conditions for the solvability of (1). These conditions enable the author to handle nearly all equations (3) with $2 \leqq m < n \leqq 50$, m and n cube-free. A table lists the cases proved impossible, others which are solvable (with a solution given, and the methods used for finding the solution), and four undecided cases. Another table lists equations of type (1) with $a < b < c$, $abc < 500$, again there being four undecided cases in addition to the solvable and nonsolvable ones. Cases are exhibited where $f \equiv 0 \pmod{p}$ is solvable for every prime p and yet (1) is not solvable, Skolem [ibid.] having established such a result for certain nonhomogeneous cubics. The methods are also applied to the equation $x^3 - 3x^2y + y^3 = 3^\lambda p z^3$, and a table contains all cases with $\lambda = 0$ or 1, $9^{1-\lambda} p \leqq 500$. On the basis of the table for (1) the author makes certain conjectures, such as: if A is an integer such that among all factorizations $A = abc$ with $a < b < c$ there is exactly one cubic f with $f \equiv 0 \pmod{p}$ solvable for every modulus, then the corresponding equation (1) is solvable.

The known impossible cases of (2) are extended as follows. Let r and q be primes with $r \equiv -q \equiv 1 \pmod{3}$, q being a cubic nonresidue of r. Then (2) has only the trivial solution with $z = 0$ in case $A \not\equiv \pm 1 \pmod{9}$ has any of the forms qr, q^2r, qr^2, q^2r^2. Again, let A be the product of three different primes, exactly one of which, say p, has the form $1 + 3k$. Then (2) has only the trivial solution if the four possible equations (1) with $abc = A$, $1 \leqq a < b < c$ can all be excluded by elementary congruence considerations mod 9 and mod p. Numerical cases are again given.

The paper is very well annotated, with clear references to other work on these equations. *I. Niven* (Eugene, Ore.).

Palamà, G. Multigrade normali del 9° ordine. Inverso del teorema di Gloden. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, [illegible] pp. (1950).

In A. Gloden's "Mehrgradige Gleichungen" [Noordhoff, Groningen, 1944; these Rev. **8**, 441] it is explained [pp. 54–55] how from a solution of a normal (ideal) multigrade equality of the 9th order a solution of

$$m^2(1089p^2 - 1053n^2) = q^2(13p^2 - 9n^2)$$

may be deduced, and conversely. J. P. Flad [Intermédiaire Recherches Math. **5**, 8 (1949)] gave numerical solutions of this last equation without indicating the manner in which he found them. The present author exposes a method of obtaining such solutions. Also a proof is given of the converse of a theorem by Gloden [loc. cit., pp. 45–46].

N. G. W. H. Beeger (Amsterdam).

Estermann, T. On sums of squares of square-free numbers. Proc. London Math. Soc. (2) **53**, 125–137 (1951).

In this paper the author considers the problem of the representation of numbers as sums of squares of square-free numbers, and by an application of the circle-method of Hardy and Littlewood establishes the following results. (1) If $s \geqq 5$ and n is a sufficiently large number for which the congruence $m_1^2 + m_2^2 + \cdots + m_s^2 \equiv n \pmod{32}$ is soluble in integers $m_1, m_2, \cdots, m_s$ not divisible by 4, then n is representable as a sum of s squares of square-free numbers. (2) If $s \geqq 8$, then any sufficiently large number is representable as a sum of s squares of square-free numbers. Result (2) is best possible in the sense that not all sufficiently large numbers can be represented as sums of 7 squares of square-free numbers. *L. Mirsky* (Bristol).

Roth, K. F. A problem in additive number theory. Proc. London Math. Soc. (2) **53**, 381–395 (1951).

It is proved that almost all positive integers u are representable in the form $u = \sum_{s=1}^{4} x_s^{s+1}$, where the x_s are positive integers; the number of exceptions in the range $1 \leqq u \leqq n$ is $O(n^{1-1/10+\epsilon})$. The proof depends on the circle method. Further it is shown elementarily that the number of integers u $(1 \leqq u \leqq n)$ representable in the form $\sum_{s=5}^{50} x_s^{s+1}$ is $> Cn^{49/51}$ $(C > 0)$, and it follows that all large numbers x can be written in the form $u = \sum_{s=1}^{50} x_s^{s+1}$. *N. G. de Bruijn* (Delft).

Bell, E. T. Solution of a functional equation in the multiplicative theory of numbers. Math. Mag. **24**, 233–235 (1951).

Let $n = \prod p_i^{a_i}$. Let $F(n)$ and $G(n)$ be multiplicative functions of the form $\prod_i b(p_i)^{a_i - 1}[b(p_i) - a(p_i)]$, where $a(p)$ and $b(p)$ are arbitrary functions. The author considers the functional equation

$$(1) \qquad \sum_{u \cdot v = n} F(u)G(v) = F(n) \cdot G(n),$$

and states that the complete solution of (1) is given by

$$(2) \quad F(n) = \prod_i \frac{f(p_i) - 1}{f(p_i) - 2}, \quad G(n) = \prod_i f(p_i)^{a_i - 1}[f(p_i) - 1],$$

where $f(x)$ is an arbitrary function, $f(p) \neq 2$ for any prime p. In (2) $F(n)$ and $G(n)$ can, of course, be interchanged.

P. Erdös (Aberdeen).

Hemer, Ove. On the highest prime-power which divides $n!$. Ark. Mat. **1**, 383–388 (1951).

The author determines the integers m with the property that p^m cannot be the highest power of p dividing $n!$ for any n. He calls these numbers the exceptional exponents of p. He shows that m is exceptional if and only if it can be written in the form

$$m = \sum_{\nu=2}^{h} a_\nu \frac{p^\nu - 1}{p - 1} - \rho,$$

where $0 \leqq a_\nu \leqq p - 1$, $a_h \neq 0$ and $1 \leqq \rho \leqq r - 1$ where $a_r \neq 0$, $a_i = 0$, $2 \leqq i < r$. Denote by $u(n)$ the number of exceptional exponents up to n. Then for sufficiently large n,

$$n/p(p-1) - \log n < u(n) < n/p(p-1).$$

P. Erdös (Aberdeen).

Ramaswami, V. Number of integers in an assigned A. P., $\leqq x$ and prime to primes greater than x^c. Proc. Amer. Math. Soc. **2**, 318–319 (1951).

Let $f(m, v, k, x, c)$ denote the number of integers of the form $m(v + nk)$, $(v, k) = 1$, $n = 1, 2, 3, \cdots$, that do not exceed x and are prime to primes greater than x^c. The case $m = v = k = 1$ was treated previously by the author [Duke Math. J. **16**, 99–109 (1949); these Rev. **10**, 597]. By suitably

generalizing the proofs used in the special case, similar formulas are found for the general case. *R. D. James.*

Selberg, Sigmund. A theorem in analytic number theory. Norske Vid. Selsk. Forh., Trondheim 23, 1–2 (1951).

Let P denote a set of primes p such that

$$\sum_{p\leqq x,\, p\in P} \frac{1}{p} > \frac{1}{h} \log\log x - c,$$

where $c>0$, $h\geqq 1$, and let $A(x;P)$ denote the number of integers $\leqq x$ which are divisible by none of the numbers of P. Then the author proves that there exists a constant D such that $A(x;P)<Dx(\log x)^{-1/h}$. The special case of the theorem in which P consists of the primes in an arithmetic progression has been applied by E. Jacobsthal [Norske Vid. Selsk. Forh., Trondheim **22**, no. 41, 193–197 (1950); these Rev. **11**, 715]. *L. Carlitz* (Durham, N. C.).

Bellman, Richard. On the functional equations of the Dirichlet series derived from Siegel modular forms. Proc. Nat. Acad. Sci. U. S. A. **37**, 84–87 (1951).

H. Maass has developed a Hecke theory of Dirichlet series satisfying functional equations for the particular case $n=2$ of Siegel's matrix modular forms [Math. Ann. **122**, 90–108 (1950); these Rev. **12**, 319]. The author sketches a new proof of the functional equation, which he states is applicable to the general case of $n\times n$ matrices. His proof follows the general lines of Riemann's second proof (involving theta-functions) of the functional equation of $\zeta(s)$. Details are promised in a future paper. *J. Lehner.*

Apostol, T. M. Identities involving the coefficients of certain Dirichlet series. Duke Math. J. **18**, 517–525 (1951).

The author proves a general theorem concerning certain Dirichlet series. Suppose that $\lambda>0$, $\kappa>0$, $\gamma=\pm1$, and let $\varphi(s)=\varphi(\sigma+it)$ satisfy the following conditions: (i) $\varphi(s)=\sum_{n=1}^{\infty} a(n)n^{-s}$ converges absolutely for $\sigma>\kappa$, (ii) $(2\pi/\lambda)^{-s}\Gamma(s)\varphi(s)=\gamma(2\pi/\lambda)^{s-\kappa}\Gamma(\kappa-s)\varphi(\kappa-s)$, (iii) $(s-\kappa)\varphi(s)$ is an integral function of finite genus. Then, if ρ is the residue of $\varphi(s)$ at $s=\kappa$,

$$\frac{1}{q!}\sum_{n=0}^{x} a(n)(x-n)^q = \rho\frac{\Gamma(\kappa)x^{\kappa+q}}{\Gamma(\kappa+q+1)} + \gamma\left(\frac{\lambda}{2\pi}\right)^q x^{\frac{1}{2}(\kappa+q)} \sum_{n=1}^{\infty} \frac{a(n)}{n^{\frac{1}{2}(\kappa+q)}} J_{\kappa+q}\left(\frac{4\pi}{\lambda}(nx)^{\frac{1}{2}}\right),$$

where $x>0$, q is a positive integer, $J_\nu(z)$ is the ordinary Bessel function of order ν, and $a(0)$ is defined to be $-\varphi(0)$. The series of Bessel functions is absolutely convergent if $q>\kappa-\frac{1}{2}$. The author has not observed that this identity is a particular case of equation (47) (p. 225) of a famous paper by E. Landau [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1915**, 209–243], Landau's function $L(x)$ reducing to a Bessel function in the author's case. The three conditions VI, VIII and IX of Landau's theorem, which is of such generality that its enunciation occupies three and a half pages, are not needed by the author, nor are they used by Landau until the second stage of his proof after (47) has been proved. A result of the same type given by A. Oppenheim [Proc. London Math. Soc. (2) **26**, 295–350 (1927)] for the sum $\sum_{n=1}^{x}(x^p-n^p)\sum_{d|n}d^{-p}$ $(0<p<\frac{1}{2})$ is proved by similar methods; here the appropriate Dirichlet series is the function $\zeta(ps)\zeta(ps+p)$. *R. A. Rankin.*

Rédei, L. A short proof of a theorem of Št. Schwarz concerning finite fields. Časopis Pěst. Mat. Fys. **75**, 211–212 (1950). (English. Czech summary)

The author gives another proof of a formula of Schwarz [Časopis Pěst. Mat. Fys. **74**, 1–16 (1949); these Rev. **11**, 328] for the number of irreducible factors of degree k of the polynomial x^m-a in a finite field whose characteristic does not divide m. *H. Davenport* (London).

Dénes, Péter. Über relativ zyklische Körper vom Primzahlgrade. Publ. Math. Debrecen **2**, 64–65 (1951).

Let l be a prime, k an algebraic number field with class number relatively prime to l, and K a cyclic extension of k of degree l. Let S be a generator of the Galois group of this extension. An ideal class A of K is called ambiguous if it is not the principal ideal class and if its (symbolic) $S-1$ power is the principal ideal class. The author proves that if there are no ambiguous ideal classes in K, then the class number of K is relatively prime to l. *W. H. Mills.*

Davenport, H. Indefinite binary quadratic forms, and Euclid's algorithm in real quadratic fields. Proc. London Math. Soc. (2) **53**, 65–82 (1951).

The author has recently written several papers on the euclidean algorithm in real quadratic fields and in cubic and biquadratic fields which have one fundamental unit. This is the first of them although others have appeared earlier. Later accounts of the quadratic case [Quart. J. Math. (2) **1**, 54–62 (1950); these Rev. **11**, 582; see also the paper reviewed below] exhibit more clearly the principle of the method which is essentially the same in the three cases. [Cf. Acta Math. **84**, 159–179 (1950); also, Trans. Amer. Math. Soc. **68**, 508–532 (1950); these Rev. **12**, 594.] *R. Hull.*

✓ ***Davenport, Harold. L'algorithme d'Euclide dans certains corps algébriques.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 41–43. Centre National de la Recherche Scientifique, Paris, 1950.

A summary account of the method and results of the papers mentioned in the preceding review, with some details in the quadratic case. *R. Hull* (Lafayette, Ind.).

Chatland, H., and Davenport, H. Euclid's algorithm in real quadratic fields. Canadian J. Math. **2**, 289–296 (1950).

The authors use a method of Davenport [see the second preceding review] to establish results (on the nonexistence of a Euclidean algorithm in six special real quadratic fields) which were previously proved at greater length by Inkeri by a method of Erdös and Ko. For the recent history of the problem, now completely solved, of determining all quadratic fields with a Euclidean algorithm, see the review of a paper by Inkeri [these Rev. **10**, 15]. For a related paper of Davenport see Quart. J. Math. Oxford. Ser. (2) **1**, 54–62 (1950); these Rev. **11**, 582. *L. Schoenfeld.*

Cugiani, M. I campi quadratici e l'algoritmo Euclideo. Period. Mat. (4) **28**, 52–62, 114–129 (1950).

An outline of the elementary arithmetical theory of quadratic number fields, with a summary of the contributions in 27 papers by various authors to the final conclusion that exactly 22 quadratic fields are Euclidean [cf. Hua's review of a paper by Inkeri, Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **41** (1947); these Rev. **10**, 15; also the preceding review]. *R. Hull* (Lafayette, Ind.).

Barnes, E. S. Non-homogeneous binary quadratic forms. Quart. J. Math., Oxford Ser. (2) 1, 199–210 (1950).

Let $f(x, y)=ax^2+bxy+cy^2$ be a real form with

$$d=b^2-4ac>0.$$

Put $\lambda=\max[|f(1,0)|, |f(0,1)|, \min(|f(1,1)|, |f(1,-1)|)]$.

Then to every pair (1) (x_0, y_0) of real numbers there is a pair (x, y) with (2) $(x, y)\equiv(x_0, y_0) \pmod 1$, (3) $|f(x, y)|\leqq\frac{1}{4}\lambda$. The sign of equality is necessary only if (1) is congruent to $(0, \frac{1}{2})$, to $(\frac{1}{2}, 0)$ or to $(\frac{1}{2}, \frac{1}{2})$. This implies a result of Davenport [Nederl. Akad. Wetensch., Proc. **49**, 815–821 (1946); **50**, 378–389, 484–491, 741–749, 909–917 (1947) = Indagationes Math. **8**, 518–524 (1946); **9**, 236–247, 290–297, 351–359, 420–428 (1947); these Rev. **8**, 444, 565; **9**, 79, 412]. Two applications are given. (I) If $f(x, y)$ is a Markoff form with the minimum Q and the discriminant $d=9Q^2-4$, (3) may be replaced by

$$|f(x, y)|\leqq\tfrac{1}{4}(2(5Q^2-4)^{\frac{1}{2}}-3Q)<\tfrac{1}{11}(2\sqrt{5}-3)d^{\frac{1}{2}}$$

for $Q\geqq5$ and by $|f(x, y)|\leqq\frac{1}{4}$, $|f(x, y)|\leqq1$ for $Q=1$, $Q=2$. This improves, for $Q>5$, a result of Davenport [loc. cit.]. (II) The existence of a Euclidean algorithm in certain real quadratic fields is proved. For the complete solution of this problem see Chatland and Davenport [see the second preceding review].

V. Jarník (Prague).

Mordell, L. J. The product of two non-homogeneous linear forms. IV. J. London Math. Soc. **26**, 93–95 (1951).

In a previous paper [same J. **18**, 218–221 (1943); these Rev. **6**, 38] the author stated a refinement of a result of Minkowski's as follows: Theorem A. Let $L=ax+by$, $M=cx+dy$ where a, b, c, d are real and $ad-bc=\Delta\neq0$. If a/b is irrational and $\epsilon>0$ is arbitrary, then for any real numbers p and q infinitely many integers (x, y) exist such that

$$(1)\quad |L+p|\cdot|M+q|<\tfrac{1}{4}|\Delta|, \qquad (3)\quad 0<|L+p|<\epsilon.$$

The author remarks that he overlooked the exceptional case when $L+p=0$ for integers x and y. In this case the author shows that under the conditions of Theorem A and for an arbitrary $\delta>0$ there exist integers x and y satisfying

$$(4)\quad |L+p|\cdot|M+q|<\Delta/\sqrt{5}+\delta, \qquad (5)\quad 0<|L+p|<\epsilon.$$

The constant $1/\sqrt{5}$ is now best possible. The author mentions the corresponding problem for n homogeneous linear forms in n variables.

B. W. Jones (Boulder, Colo.).

Roth, K. F. On a problem of Heilbronn. J. London Math. Soc. **26**, 198–204 (1951).

Given a convex domain K of area m, let $D(n)$ run through the n-tuples of mutually distinct points $p_1, p_2, \cdots, p_n$ in K and put

$$\Delta_n(K)=m^{-1}\sup_{D(n)}\,(\min_{x<y<z}\Delta P_xP_yP_z),$$

where $\triangle ABC$ denotes the area of the triangle ABC. Obviously $\Delta_n(K)=O(n^{-1})$. Heilbronn conjectured $\Delta_n(K)=O(n^{-2})$. The author proves $\Delta_n(K)=O(1/n(\log\log n)^{\frac{1}{2}})$.

P. Scherk (Saskatoon, Sask.).

LeVeque, William J. Note on the transcendence of certain series. Proc. Amer. Math. Soc. **2**, 401–403 (1951).

The series studied by the author have the form

$$\sum_{n=0}^{\infty}(n+k)^{n+m}x^n/n!,$$

where k and m are integers. It is shown that the sum of such a series is a transcendental number for every algebraic value of x with $0<|x|<e^{-1}$ in each of the following two cases: (a) $k=0$ or 1 and $m\geqq0$, (b) $k\neq0$, 1 and $m\geqq-1$. (0^0 is interpreted as 1.) The proof depends on the well-known theorem, that e^α is transcendental for every algebraic value of $\alpha\neq0$.

J. Popken (Utrecht).

Slater, N. B. The distribution of the integers N for which $\{\theta N\}<\phi$. Proc. Cambridge Philos. Soc. **46**, 525–534 (1950).

In Verschärfung des Weylschen Theorems, dass $\{\theta N\}$ für irrationales θ gleichverteilt ist (mod 1), wird folgendes untersucht: Wie gross sind die Lücken, die zwischen aufeinanderfolgenden ganzen Zahlen N liegen, falls N der Ungleichung $\{\theta N\}<\phi$ genügt, wobei $\{\theta N\}$ den positiven Rest von θN mod 1 bedeutet? Wie häufig treten die verschiedenen Lücken dabei auf? Diese Fragen werden unter Verwendung der Kettenbruchdarstellung von θ in der Weise beantwortet, dass die Gestalt der möglichen Lücken angegeben und für endliche N auch deren Anzahl mitgeteilt wird, während für beliebig grosse N ein asymptotischer Wert bewiesen wird. Die Untersuchung wird für rationales und irrationales θ getrennt geführt. Die Beweise sind völlig elementar und nicht tief liegend.

T. Schneider (Göttingen).

ANALYSIS

Tsuboi, Chuji. A problem of weighted mean. Bull. Earthquake Res. Inst. Tokyo **19**, 458–475 (1941). (English. Japanese summary)

Given functions $\psi(x)$ and $\phi(x)$ and the relation

$$\psi(x)=\int_0^b\phi(x+z)p(z)dz,$$

the problem of solving for the function $p(z)$ is considered. It is shown that if all three functions are expansible into Fourier series, the Fourier coefficients of the mth order of $p(z)$ are simple algebraic functions of those of $\psi(x)$ and $\phi(x)$ of the corresponding order. Several examples of the application of the method to observational data are given.

T. N. E. Greville (Washington, D. C.).

Redheffer, R. M., and Steinberg, R. The Laplacian and mean values. Quart. Appl. Math. **9**, 315–317 (1951).

Let $A(f, P, r)$ denote the mean value of the function f over the surface of the sphere of radius r and center P. It is well known that the ratio $6(A(f, P, r)-f(P))/r^2$ tends to the value of the Laplacian $\nabla^2f(P)$ as $r\rightarrow0$, if f is sufficiently smooth. The authors wish to characterize the point sets for which a similar relation holds, and obtain the following result: Let P be a fixed point on the bounded point set S on which a measure m is defined such that $0<m(S)=M<\infty$. Then (using vectorial notation)

$$\int_S f(P+t(Q-P))dm(Q)-Mf(P)\sim(t/2)^2I\nabla^2f(P) \quad \text{as} \quad t\rightarrow0$$

for an arbitrary function f having continuous third derivatives near P if and only if S has its center of mass at P and its moment of inertia about every axis through P is I.

W. Rudin (Cambridge, Mass.).

Mihlin, S. G. On some estimates connected with Green's functions. Doklady Akad. Nauk SSSR (N.S.) **78**, 443–446 (1951). (Russian)

Let $u(x_1, x_2)$ be a real-valued continuous function defined on the closed unit circle $x_1^2+x_2^2\leqq1$, which van-

ishes on the boundary $x_1^2+x_2^2=1$, and whose Laplacian $\partial^2u/\partial x_1^2+\partial^2u/\partial x_2^2$ is of integrable square on $x_1^2+x_2^2<1$. It is shown that there exists a real number C such that the inequality

$$\iint_{x_1^2+x_2^2<1}\left|\frac{\partial^2 u}{\partial x_k\partial x_m}\right|^2 dx_1dx_2 \leqq C^2 \iint_{x_1^2+x_2^2<1}\left|\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}\right|^2 dx_1dx_2,$$

where $k=1$ or 2 and $m=1$ or 2, holds for any such function u. The proof makes use of a formula of F. Tricomi [Math. Z. 27, 87–133 (1927)] for differentiating Cauchy principal value integrals and of a previous theorem of the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) 15, 429–432 (1937); Uspehi Matem. Nauk 3, no. 3(25), 29–112 (1948); translated as Amer. Math. Soc. Translation no. 24 (1950); these Rev. 10, 305; 12, 107] concerning integrals of this same type.

J. B. Diaz (College Park, Md.).

Olovyanišnikov, V. M. On inequalities between upper bounds of consecutive derivatives on a half-line. Uspehi Matem. Nauk (N.S.) 6, no. 2(42), 167–170 (1951). (Russian)

The author obtains the best possible bound for M_k, the maximum of $f^{(k)}(x)$ on $(-\infty, 0)$, in terms of M_0 and M_n, under the assumption that $f(x), \cdots, f^{(n)}(x)$ are all nondecreasing on $(-\infty, 0)$. His result is

$$M_k^n \leqq (n!)^{n-k}\{(n-k)!\}^{-n}M_0^{n-k}M_n^k.$$

Equality is attained for

$$\varphi_n(x)=0,\ -\infty<x<-l;\quad \varphi_n(x)=a(l+x)^n/n!,\ -l\leqq x\leqq 0.$$

The author proves the theorem first for $k=1$ and extends it to general k by the method used by Kolmogorov for $(-\infty, \infty)$ [Uchenye Zapiski Moskov. Gos. Univ. Matematika 30, 3–16 (1939); Amer. Math. Soc. Translation no. 4 (1949); these Rev. 1, 298; 11, 86]. *R. P. Boas, Jr.*

Boas, R. P., Jr., and Chandrasekharan, K. Addendum: Derivatives of infinite order. Proc. Amer. Math. Soc. 2, 422 (1951).

The authors point out that a result of T. Bang [Thesis, Copenhagen, 1946; these Rev. 8, 199] contains a result which is needed to complete the proof of one of their theorems [Bull. Amer. Math. Soc. 54, 523–526 (1948); these Rev. 10, 21]. *S. Mandelbrojt.*

Aissen, Michael, Edrei, Albert, Schoenberg, I. J., and Whitney, Anne. On the generating functions of totally positive sequences. Proc. Nat. Acad. Sci. U. S. A. 37, 303–307 (1951).

The authors announce the following solution of a problem of Schoenberg. The sequence $\{a_n\}_{n=0}^{\infty}$, $a_0=1$, is totally positive (i.e., all minors of the infinite matrix $\|a_{i-k}\|$, $a_n=0$ for $n<0$, are nonnegative) if and only if the generating function $f(z)=\sum_{n=0}^{\infty}a_nz^n$ has the form $e^{\gamma z}\prod_{\nu=1}^{\infty}(1+\alpha_\nu z)/\prod_{\nu=1}^{\infty}(1-\beta_\nu z)$ with all parameters nonnegative and $\sum\alpha_\nu$, $\sum\beta_\nu$ convergent. Several applications are given to polynomials and entire functions with real negative zeros. Proofs are to appear in later publications by various subsets of the set of authors.

R. P. Boas, Jr. (Evanston, Ill.).

Tomić, M. On some theorems of Fejér and Szegö concerning Taylor series with multiply monotone coefficients. Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka 198, 175–185 (1950). (Serbo-Croatian)

This is a rather condensed version of a paper which also appeared in French [Acad. Serbe Sci. Publ. Inst. Math. 3, 243–258 (1950); these Rev. 12, 813]. It contains one theorem which does not appear in the French version: if the sequence $\{c_n\}$ is quadruply monotone and $R_n(z)=\sum_{k=n}^{\infty}c_kz^k$, then

$$|R_{n-1}(z)|-2|R_n(z)|+|R_{n+1}(z)|\geqq 0;$$

i.e., $\{|R_n|\}$ is convex. The hypothesis is more restrictive but the conclusion is stronger than in Fejér's theorem [Math. Z. 24, 267–284 (1925)] that if $\{c_n\}$ is triply monotone then $\{|R_n|^2\}$ is convex. Unfortunately, the proof of the theorem is missing from the paper, having been deleted by an editorial oversight, but the author has communicated it to the reviewer. The essential idea is to consider the series whose coefficients are Δ^3c_n and apply theorems of Fejér [Trans. Amer. Math. Soc. 39, 18–59 (1936)] on trigonometric series with convex coefficients. *R. P. Boas, Jr.*

Timan, A. F. On quasi-smooth functions. Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, 243–254 (1951). (Russian)

The main result of the paper is a complete proof of a theorem announced previously [Doklady Akad. Nauk SSSR (N.S.) 70, 961–963 (1950); these Rev. 11, 422; in the review the condition was omitted that the function $f(x)$ was to vanish at the endpoints of the interval (a, b)]. It is also shown that if $f(x)$, $-1\leqq x\leqq +1$, is continuous and satisfies the condition

$$\left|f(x_1)-2f\left(\frac{x_1+x_2}{2}\right)+f(x_2)\right|\leqq M|x_1-x_2|,$$

then the best approximation $E_n[f]$ of f by polynomials of degree $\leqq n$ satisfies the inequality $E_n[f]=O(1/n)$. This is an analogue of a result well known for periodic functions [see Zygmund, Duke Math. J. 12, 47–76 (1945); these Rev. 7, 60]. *A. Zygmund* (Chicago, Ill.).

Lorentz, G. G. Deferred Bernstein polynomials. Proc. Amer. Math. Soc. 2, 72–76 (1951).

If $f(x)$ is continuous for $0\leqq x\leqq 1$ it is well known that its sequence of Bernstein polynomials

$$B_n(f;x)=\sum_{k=0}^{n}f(k/n)\binom{n}{k}x^k(1-x)^{n-k}$$

converges to $f(x)$ uniformly for $0\leqq x\leqq 1$. The behavior of $B_n(f;x)$ under various discontinuity assumptions on f has also been investigated in some detail, for instance, by Chlodovsky [Fund. Math. 13, 62–72 (1929)] and by Herzog and Hill [Amer. J. Math. 68, 109–124 (1946); these Rev. 7, 440]. In the present paper the author makes a further contribution in this direction. He introduces the "deferred Bernstein polynomials" $B_n(f-a;x)$ and obtains the following results. (i) Let $f(x)$ be a function of period 1 which is continuous for $0<x<1$ and such that $|f(x)|=O[\exp(|x|^{-1/(2+\epsilon)})]$ as $x\to 0$ for some $\epsilon>0$. Then $B_n(f-a;x)\to f(x-a)$ is valid for almost all a and all $x\neq a$. (ii) Under the hypotheses of (i), if $f(x)\to\infty$ as $x\to 0$, then there is a set of the a's having the power of the continuum, such that the sequence $B_n(f-a;x)$ is unbounded for any $x\neq a$, 0, 1. In (i) and (ii) the degree of transcendency of a plays an important role. The author next observes that for any integrable f the relation $B_{n_k}(f-a;x)\to f(x-a)$ holds for almost all a and x if $n_k\to\infty$ is a properly chosen sequence depending on f. Such a sequence n_k, independent of f, is shown to exist in case $\sum_{m\neq 0}c_m^2\log|m|<+\infty$, where the c_m are the complex Fourier coefficients of f. *J. D. Hill.*

Hummel, P. M., and Seebeck, C. L., Jr. A new interpolation formula. Amer. Math. Monthly 58, 383–389 (1951).

Let $f(x)$, together with its first $m+n+1$ derivatives, be continuous in an interval containing $x_0, x_1, \cdots, x_n$, where $x_i \neq x_j$. It is shown that

$$f(x)=\sum_{i=0}^{n}\prod_{j\neq i}\frac{x_j-x}{x_j-x_i}S_i+R,$$

where

$$S_i=f(x_i)+\sum_{k=1}^{m}c_k'f^{(k)}(x_i)(x-x_i)^k,$$
$$c_k'=(m+n-k)!m!/(m+n)!(m-k)!k!,$$
$$R=\sum_{i=0}^{n}\prod_{j\neq i}(x_j-x)(x_j-x_i)^{-1}R_i,$$

and

$$R_i=(-1)^n[m!n!/(m+n)!(m+n+1)!]\times(x-x_i)^{m+n+1}f^{(m+n+1)}(\theta_i),$$

where θ_i is between x and x_i. This formula is used to obtain a correction for linear interpolation (when $n=m=1$).

T. N. E. Greville (Washington, D. C.).

Barna, Béla. Über das Newtonsche Verfahren zur Annäherung von Wurzeln algebraischer Gleichungen. Publ. Math. Debrecen 2, 50–63 (1951).

A short while ago A. Rényi [Mat. Lapok 1, 278–293 (1950); these Rev. 12, 321] studied the influence of the choice of the first approximation on the convergence of Newton's method. He obtained a result for a function with exactly three real roots which has a continuous first and a monotone increasing and continuous second derivative. Rényi also raised two questions [see the review cited above] concerning the points of divergence of the iteration. The author considers the same aspect of Newton's method and discusses polynomials of degree four with exactly four different real roots. He classifies the points of divergence of the iteration and also shows that the set of points of divergence is not enumerable. Thus he solves Rényi's first problem.

E. Lukacs (Washington, D. C.).

Calculus

✶Kuntzmann, J. Analyse appliquée. 1re partie. Centre de Documentation Universitaire, Paris, undated. 348+14 pp.

A textbook aimed at supplying the most important mathematical techniques for engineers, physicists, etc. A rigorous development is not attempted. However, some of the vocabulary usually associated with such a development is introduced intuitively and then used. About a third of the book is devoted to definitions and properties of integrals and series and special functions defined by integrals, another third to vectors, fields and potentials, and the rest to functions of a complex variable. The book is photolithoprinted from a typed manuscript.

J. V. Wehausen.

✶Hofmann, August. Einführung in die Vektorrechnung. Verlag Von R. Oldenbourg, München, 1951. 107 pp.

An introductory textbook for use in schools.

L. M. Milne-Thomson (Greenwich).

Agnew, Ralph Palmer. Mean values and Frullani integrals. Proc. Amer. Math. Soc. 2, 237–241 (1951).

Let

$$I(a,b)=\int_0^\infty[f(at)-f(bt)]t^{-1}dt = \lim_{\epsilon\to0,\,h\to\infty}\int_\epsilon^h[f(at)-f(bt)]t^{-1}dt,$$

with $a, b>0$, $f(t)$ Lebesgue integrable over $0<m<t<M<\infty$. Then

$$I(a,b)=\lim_{h\to\infty}\int_{bh}^{ah}f(t)t^{-1}dt-\lim_{\epsilon\to0}\int_{b\epsilon}^{a\epsilon}f(t)t^{-1}dt$$

when the limits exist. If in the last two integrals, respectively, we set $t=e^u$, $t=e^{-u}$ and $\lambda=\log(a/b)$, then

$$I(a,b)=\lim_{A\to\infty}\int_A^{\lambda+A}f(e^u)du+\lim_{B\to\infty}\int_B^{-\lambda+B}f(e^{-u})du$$

when the limits exist. This gives the Frullani formula

$$I(a,b)=\lambda[\lim_{x\to\infty}f(x)-\lim_{x\to0}f(x)]$$

when the limits exist. The author gives simple proofs of the following in which $f(x)$ is Lebesgue integrable on $0<a<x<b<\infty$. If one of the mean values

$$\lim_{x\to\infty}x^{-1}\int_1^x f(t)dt=\lim_{x\to\infty}x\int_x^\infty f(t)t^{-2}dt$$

exists then both exist and the inequality holds. If one of the mean values

$$\lim_{x\to0}x^{-1}\int_0^x f(t)dt=\lim_{x\to0}x\int_x^1 f(t)t^{-2}dt$$

exists then both exist and the inequality holds. If $f(t)$ is Lebesgue integrable over each finite interval and the left member of

$$\int_0^\infty[f(at)-f(bt)]t^{-1}dt=\lambda\left[\lim_{x\to\infty}x^{-1}\int_0^x f(t)dt-\lim_{x\to0}x^{-1}\int_0^x f(t)dt\right]$$

exists for each $\lambda=\log[a/b]$ in a set of positive measure then the mean values in the right member exist for each pair of positive numbers a and b. If the mean values in the right member exist then the left member exists for each pair of positive numbers a and b.

R. L. Jeffrey.

Ostrowski, Alexander M. Note on an infinite integral. Duke Math. J. 18, 355–359 (1951).

The author's Theorem I is: Let $f(x)$ be absolutely continuous for $0<x<\infty$ and for an $\alpha>0$ let

$$\int_0^\infty x^{-\alpha}f'(x)dx=\lim\int_\epsilon^A x^{-\alpha}f'(x)dx,\quad(\epsilon\to0,\ A\to\infty)$$

exist. Then $f(+0)=\lim_{x\to0}f(x)$ exists and

$$\int_0^\infty x^{-\alpha}f'(x)dx=\alpha\int_0^\infty x^{-(\alpha+1)}[f(x)-f(+0)]dx.$$

For the case $\alpha=1$ there is a formula by A. Winckler [Akad. Wiss. Wien, S.-B. 60 (1869), 857–917 (1870)] which is

$$\int_0^\infty x^{-2}[f(bx)-f(ax)]dx=(b-a)\int_0^\infty x^{-1}f'(x)dx$$

which has been considered by J. Bertrand [Traité de calcul différentiel et de calcul intégral, vol. 2, Gauthier-Villars, Paris, 1870] and G. Frullani [Memorie di Matematica della

Società Italiana delle Scienze Residente in Modena **20**, 448–467 (1830)], and the author says that in both cases the results are wrong. Theorem II of the present note is: For positive p, α, a and b let

$$\int_p^\infty x^{-(\alpha+1)}[f(ax)-f(bx)]dx=\lim_{A\to\infty}\int_p^A x^{-(\alpha+1)}[f(ax)-f(bx)]dx$$

exist, where $f(x)$ is integrable on (p, x), $x<\infty$. Then $\int_p^\infty x^{-(\alpha+1)}f(x)dx$ exists. *R. L. Jeffrey* (Kingston, Ont.).

Neville, E. H. A trigonometrical inequality. Proc. Cambridge Philos. Soc. **47**, 629–632 (1951).

By an elementary argument the author shows that

$$\csc^2 x-(2m+1)^{-1}<\sum_{r=-m}^{m}(x-r\pi)^2<\csc^2 x;$$

hence he obtains a short and elementary evaluation of $\int_0^\infty x^{-1}\sin x dx$, the infinite product for $\sin x$ and the infinite series for $\csc x$, from which again the integral is readily evaluated. *R. P. Boas, Jr.* (Evanston, Ill.).

Theory of Sets, Theory of Functions of Real Variables

Sierpiński, W. Sur les types ordinaux des ensembles linéaires. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 427–428 (1950).

Let (φ, ψ) mean that every ordered set of type φ is similar to a subset of an ordered set of type ψ. If (φ, ψ) but not (ψ, φ), φ is said to be smaller than ψ $(\varphi<\psi)$. If neither (φ, ψ) nor (ψ, φ), φ and ψ are said to be incomparable [cf. Fraïssé, C. R. Acad. Sci. Paris **226**, 1330–1331 (1948); these Rev. **10**, 517]. Let λ denote the order type of the set of real numbers in their natural order. The author announces that he has obtained the following results with the aid of the axiom of choice [proofs are in the paper reviewed below]: (1) If $\varphi_1, \varphi_2, \cdots$ is an infinite sequence of order types with $\varphi_i<\lambda$, $i=1, 2, \cdots$, then there exists an order type ψ such that $\varphi_i<\psi<\lambda$, $i=1, 2, \cdots$; (2) there exist two order types, φ_1 and φ_2, having the power of the continuum, such that $\varphi_1<\varphi_2<\lambda$, and such that there exists no order type ψ with $\varphi_1<\psi<\varphi_2$; (3) if φ is an order type which has the power of the continuum and is smaller than λ, then there exists an order type ψ which has the power of the continuum and is smaller than φ; (4) if the continuum hypothesis is true, then there exist two nonenumerable types $\varphi_1<\lambda$ and $\varphi_2<\lambda$ such that there is no nonenumerable type ψ with $\psi<\varphi_1$ and $\psi<\varphi_2$; (5) there exists an increasing sequence (and also a decreasing sequence) having the power of the continuum, whose terms are order types of linear sets having the power of the continuum; (6) there exists a class of $2^{\aleph}$ order types of linear sets having the power of the continuum, such that every two types in this class are incomparable.

F. Bagemihl (Rochester, N. Y.).

Sierpiński, Wacław. Sur les types d'ordre des ensembles linéaires. Fund. Math. **37**, 253–264 (1950).

Proofs are given of the results stated in the preceding review, and of several related theorems. *F. Bagemihl.*

Fodor, G., and Ketskeméty, I. Some theorems on the theory of sets. Fund. Math. **37**, 249–250 (1950).

In connection with some problems concerning relations [cf. Sierpiński, Fund. Math. **28**, 71–74 (1937) and S. Piccard, ibid. **28**, 197–202 (1937); **29**, 5–8 (1937)], the authors have established the following results: Let E be a nonvoid set and R a binary relation between $x\epsilon E$ and nonvoid $Y\subseteq E$, such that for every nonvoid $Y\subseteq E$ there is a single $x\epsilon Y$ satisfying xRY (consequently, for any $x\epsilon E$ and any nonvoid $Y\subseteq E$, the relation xRY is either true or false). If n is a cardinal $\leqq kE$ (=cardinal of E), let E_n be the set of all $x\epsilon E$ each of which is R-connected with $\leqq n$ subsets of E; then $2^z-1\leqq nz$, $z=kE_n$ (theorem I). Let $n<kE$; denote by F_n the set of all $x\epsilon E$ each of which is R-connected only with $Y\subseteq E$ such that $kY<n$; then $kF_n\leqq n$ (theorem III). The results are very closely connected with the results obtained by the same authors in Portugaliae Math. **9**, 145–147 (1950); these Rev. **12**, 809. *Đ. Kurepa* (Zagreb).

Choquet, Gustave. Ensembles boréliens et analytiques dans les espaces topologiques. C. R. Acad. Sci. Paris **232**, 2174–2176 (1951).

Alors que les ensembles boréliens et analytiques relatifs à un espace E sont définis en partant des ensembles fermés (ouverts) de E, le point de départ de l'auteur c'est la classe $K=K_0$ des compacts$\subseteq E$; par induction l'auteur définit les classes K_α $(\alpha<\omega_1)$ des "ensembles K-boréliens" comme intersections (réunions) denombrables d'éléments extraits des K_β $(\beta<\alpha)$ si α est pair (impair); en particulier, $K_1=K_\sigma$, $K_2=K_{\sigma\delta}$. La classe $K_{\sigma\delta}$ est bien importante et joue le rôle de la classe des G_δ. Les ensembles analytiques relatifs à un espace séparé E sont définis comme images continues dans E d'un $K_{\sigma\delta}$ extrait d'un espace compact [cf. aussi N. Lusin, Ensembles analytiques . . . , Gauthier-Villars, Paris 1930, p. 134]. Ainsi, tout ensemble K-borélien d'un espace séparé est analytique. Le problème de l'invariance topologique des classes K_α reste ouvert; même, le problème de savoir si l'homéomorphe d'un K-borélien est encore K-borélien reste ouvert (les problèmes analogues pour des ensembles boréliens sont résolus par l'affirmative [cf. C. Kuratowski, Topologie I, Warszawa-Lwów, 1933, p. 217]). L'auteur pose d'autres définitions (ensembles d'unicité, ensembles K-boréliens généralisés, etc.) et énonce quelques résultats.

Đ. Kurepa (Zagreb).

Pettis, B. J. On the extension of measures. Ann. of Math. (2) **54**, 186–197 (1951).

Let X be a sublattice of a boolean algebra A, and let φ be a function on X to an abelian group G. It is shown that φ admits an additive extension on the ring generated by X if and only if (1) $\varphi(x\vee y)+\varphi(x\wedge y)=\varphi(x)+\varphi(y)$, and (2) $\varphi(0)=0$ in case $0\epsilon X$. If A is σ-complete and G is the reals, then φ can be extended to a σ-finite measure on the σ-ring generated by X provided, in addition, (3) φ is nonnegative, (4) monotone increasing, and (5) satisfies an approximation condition which serves to justify an abstract form of the reasoning ordinarily based on compactness. Conditions under which a function defined on a lattice of closed or open subsets of a topological space can be extended to a regular measure are obtained and applied to a proof of the Riesz-Markoff theorem and to extension of the content functions that arise in defining Haar measure. *J. C. Oxtoby.*

Schmidt, Robert. Zur Orthogonalinvarianz des Inhalts. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1950**, 103–106 (1951).

A new proof is given for the fact that the inner and outer contents of the sets in a Euclidean space E_k are invariant under orthogonal transformations. *A. Rosenthal.*

Očan, Yu. S. A generalized integral. Mat. Sbornik N.S. **28**(70), 293–336 (1951). (Russian)

Let E be a measurable linear set of positive finite measure. For a measurable, non-negative, real-valued function φ defined on E and real numbers $t\geqq 0$, let φ_t (the truncated function) be min (φ, t). Let $f(t)$ be a positive function defined for $t\geqq 0$ such that f is non-decreasing and such that (A) $\lim_{t\to\infty} f(at)/f(t)=1$ for all $a>0$. The author's generalized integral rests upon the following definitions (in the original a G clef sign is used in place of $\mathfrak{S}$):

$$\text{(B)}\qquad (f(t))\overline{\mathfrak{S}}\varphi(x)dx=\limsup_{t\to\infty}\frac{\int_E \varphi_t(x)dx}{f(t)};$$

$$\text{(C)}\qquad (f(t))\underline{\mathfrak{S}}\varphi(x)dx=\liminf_{t\to\infty}\frac{\int_E \varphi_t(x)dx}{f(t)};$$

$$\text{(D)}\qquad (f(t))\mathfrak{S}\varphi(x)dx=\lim_{t\to\infty}\frac{\int_E \varphi_t(x)dx}{f(t)}.$$

The motivation for these definitions, of course, is to obtain a functional which is finite for a class of measurable functions with Lebesgue integral $+\infty$. The function φ is said to be upper, lower, or exactly integrable with modulus $f(t)$ if the limits (B), (C), and (D), respectively, exist and are finite. The first part of the paper is devoted to a study of the functionals (B), (C), and (D) for non-negative functions φ. For example, $\overline{\mathfrak{S}}(\varphi+\psi)dx\leqq\overline{\mathfrak{S}}\varphi dx+\overline{\mathfrak{S}}\psi dx$; $\underline{\mathfrak{S}}(\varphi+\psi)dx\geqq\underline{\mathfrak{S}}\varphi dx+\underline{\mathfrak{S}}\psi dx$; $\mathfrak{S}(\varphi+\psi)dx=\mathfrak{S}\varphi dx+\mathfrak{S}\psi dx$ if any two of the functions are integrable; and so on. It is shown that $\mathfrak{S}\varphi dx$ is a finitely but usually not countably additive function of the set E. It is also shown that a function may be integrable on a set E and be non-integrable on a measurable subset of E.

The author takes up next the problem of integrating sequences of functions term by term. It is apparent that Lebesgue's theorem on term-by-term integration is false for the integral $\mathfrak{S}$. A restriction sufficient for several purposes is provided as follows. A family $\{\varphi(x, \alpha)\}_{\alpha\epsilon A}$ of non-negative functions is said to be semi-uniformly upper integrable if for every $\epsilon>0$ there exists a $T(\epsilon)$ such that for all $\alpha\epsilon A$ and all $t>T$,

$$\frac{1}{f(t)}\int[\varphi(x,\alpha)]_t dx\leqq\overline{\mathfrak{S}}\varphi(x,\alpha)dx+\epsilon.$$

A similar definition is given for semi-uniform integrability. An analogue of Fatou's theorem is proved: if $\{\varphi_n(x)\}_{n=1}^{\infty}$ are non-negative, converge pointwise to $F(x)$, and are semi-uniformly upper integrable, then

$$\overline{\mathfrak{S}}F(x)dx\leqq\lim_{n\to\infty}\overline{\mathfrak{S}}\varphi_n(x)dx.$$

In the same fashion, if $\{\varphi_n(x)\}_{n=1}^{\infty}$ are non-negative, are semi-uniformly integrable and converge pointwise to the function $F(x)$, and if $F(x)$ is integrable, then

$$\mathfrak{S}F(x)dx=\lim_{n\to\infty}\mathfrak{S}\varphi_n(x)dx.$$

A restriction stronger than semi-uniform integrability is embodied in the following definition: a family $\{\varphi(x,\alpha)\}_{\alpha\epsilon A}$ of non-negative functions is uniformly integrable if for every $\epsilon>0$, there exists a $T(\epsilon)>0$ such that for all $\alpha\epsilon A$ and $t>T(\epsilon)$,

$$\text{(E)}\qquad \left|\mathfrak{S}\varphi(x,\alpha)dx-\frac{1}{f(t)}\int_E[\varphi(x,\alpha)]_t dx\right|<\epsilon.$$

The principal result here is that if $\{\varphi_n(x)\}_{n=1}^{\infty}$ is a sequence of uniformly integrable functions with pointwise limit $\psi(x)$, then ψ is integrable and $\lim_{n\to\infty}\mathfrak{S}\varphi_n(x)dx=\mathfrak{S}\psi(x)dx$. A restricted form of Fubini's theorem and of Hölder's and Minkowski's inequalities are also established.

Up to this point, all functions integrated have been non-negative. Noting that the usual definition of the integral for functions of variable sign results here in a non-linear functional, the author adopts a different device. Let $\varphi_+=\max(\varphi, 0)$ and $\varphi_-=-\min(\varphi, 0)$. If the expression

$$\text{(F)}\qquad \frac{1}{f(t)}\left[\int[\varphi_+(x)]_m dx-\int[\varphi_-(x)]_n dx\right]$$

has a lim sup, lim inf, or lim as $t\to\infty$ which is independent of the arbitrary positive numbers m and n, then the lim sup, lim inf, and lim so obtained are defined as the upper integral, lower integral, and integral, respectively, of φ, and are designated, as above, by the symbols $\overline{\mathfrak{S}}$, $\underline{\mathfrak{S}}$, and $\mathfrak{S}$. The usual properties are proved for these functionals. In particular, it is shown that if $\varphi+\psi$ are integrable and a and b are any real numbers, then $a\varphi+b\psi$ is integrable and

$$\mathfrak{S}[a\varphi(x)+b\psi(x)]dx=a\mathfrak{S}\varphi(x)dx+b\mathfrak{S}\psi(x)dx.$$

If the function $f(t)$ is identically 1, then the integral defined by (F) is actually an extension of the Lebesgue integral. The last part of the paper investigates the relation between this extended integral and the integrals of Denjoy and Boks [Rend. Circ. Mat. Palermo **45**, 211–264 (1921)] and the generalized mean value of Kolmogoroff [Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer, Berlin, 1933].

E. Hewitt (Uppsala).

Henstock, R. Density integration. Proc. London Math. Soc. (2) **53**, 192–211 (1951).

Ce mémoire contient plusieurs définitions et de nombreux résultats relatifs aux fonctions d'intervalles. Au §1, l'auteur dit pourquoi il trouve insuffisantes les notions introduites jusqu'ici. Voici une analyse assez détaillée du contenu des autres § qui sont relativement indépendants.

§2. Soit $g(I)$ une fonction d'intervalle définie pour tout $I\epsilon W=[0, 1]$, $g_{\underline{B}}(R)$ et $g_{\overline{B}}(R)$ les intégrales inf. et sup. de Burkill de g (R=réunion finie d'intervalles), $g_B(R)$ leur valeur commune éventuelle [voir, par exemple, R. Henstock, J. London Math. Soc. **21**, 204–209 (1946); ces Rev. **8**, 572]. Pour tout E mesurable, l'auteur pose $K(I)=g(I)[m(EI)/m(I)]$ et nomme "density integrals" inf. et sup. de g sur E les quantités $g_{\underline{D}}(E)=K_{\underline{B}}(W)$ et $g_{\overline{D}}(E)=K_{\overline{B}}(W)$ (en modifiant les notations de l'auteur pour abréger). Si elles sont égales, il dit que la "density integral" $g_D(E)$ existe. Il démontre les résultats suivants: (1) Si E est un R, $g_{\underline{D}}(R)\leqq g_{\underline{B}}(R)\leqq g_{\overline{B}}(R)\leqq g_{\overline{D}}(R)$. Si en outre g est continue, les deux inégalités extrêmes deviennent des égalités. (2) (Théorème principal). Pour que $g_D(E)$ existe, quel que soit E mesurable, il faut et il suffit que $g_B(W)$ existe et que g soit absolument continue. En ce cas, $g'(x)=\lim g(I)/m(I)$ existe presque partout et $g_D(E)=\int_E g'(x)dx$. Il indique le résultat suivant obtenu dans sa thèse (non encore publiée): pour que $g_{\underline{D}}(E)$ et $g_{\overline{D}}(E)$ soient finis quel que soit E, il faut et il suffit que g soit à variation bornée. Il donne exemple

où G est une réunion d'intervalles I_i, g est continue, et $g_{\overline{B}}(G) \neq \sum_1^\infty g_{\overline{B}}(I_i)$. Il indique des relations entre la présente notion et la "k-intégration" qu'il a introduite antérieurement [loc. cit.].

§3. (1) Supposons $f(x)$ bornée et intégrable sur W. Posons $K(I)=[g(I)/m(I)]\int_I f dx$. La "density integral" inf. de f par rapport à g est par définition $(D)\underline{\int} fdg = K_{\underline{B}}(W)$; de même pour la "density integral" sup. Si elles sont égales, la "density integral" $(D)\int fdg$ existe. Soit $\lambda(I)$ la plus grande des bornes inférieures de $f(\xi)g(I)$ pour presque tous les ξ dans I et $\Lambda(I)$ la plus petite des bornes supérieures. Alors: 1. $\lambda_{\underline{B}}(W) \leqq (D)\underline{\int} fdg \leqq (D)\overline{\int} fdg \leqq \Lambda_{\overline{B}}(W)$.

2. Supposons f continue sauf sur un ensemble où la variation de g est nulle, et g à variation bornée. En ce cas $\lambda_{\underline{B}}(W)=\Lambda_{\underline{B}}(W)=(D)\underline{\int} fdg$, $\lambda_{\overline{B}}(W)=\Lambda_{\overline{B}}(W)=(D)\overline{\int} fdg$.

3. Supposons que $g_B(I)$ existe pour tout I. Alors $\underline{\int} fdg_B \leqq (D)\underline{\int} fdg \leqq (D)\overline{\int} fdg \leqq \overline{\int} fdg_B$, les intégrales extrêmes étant prises au sens de Riemann-Stieltjes. De plus, (a) si les conditions de (2) (ci-dessus) sont remplies, $(D)\int fdg$ existe et est égale à $\int fdg_B$, (b) si g est absolument continue $(D)\int fdg$ existe et est égale à $\int fg'dx$. (2) On dit que f vérifie la condition $\mathfrak{T}$ ou est quasicontinue par rapport à g si pour tout ϵ il existe $f_1(x)$ continue, égale à f sauf dans un ouvert G de mesure $<\epsilon$ sur lequel la variation de g est $<\epsilon$, et linéaire dans les intervalles de G; soit $\lambda(I)$ la fonction définie comme ci-dessus correspondant à f_1; alors, par définition, $\liminf_{\epsilon\to 0} \lambda_{\underline{B}}(W)=(\mathfrak{TS})\underline{\int} fdg$ sera l'intégrale inf. de Tonelli-Stieltjes de f par rapport à g; de même pour l'integrale sup. Si elles sont égales, l'auteur dit que l'intégrale $(\mathfrak{TS})\int fdg$ existe. Si f est mesurable B, si g est à variation bornée et si g_B existe, cette notion coïncide avec l'intégrale de Lebesgue-Stieltjes, mais non avec les "density integrals" inf. et sup. (3) Extensions au cas où f n'est pas bornée. Exemple dans lequel g est additive et absolument continue, f et fg' sommables, et $(D)\int fdg \neq \int fdg$, cette dernière prise au sens de Lebesgue-Stieltjes.

§4. Soit $\gamma(E)$ une fonction d'ensemble additive au sens restreint définie sur une famille F d'ensembles de W. On en déduit une fonctionnelle linéaire $P(\varphi)$ définie sur les combinaisons linéaires φ de fonctions caractéristiques d'ensembles de F. Peut-on la prolonger par continuité dans l'espace des fonctions f de carré sommable? L'auteur démontre que, pour cela il faut et il suffit que la variation de $\gamma^2(E)/m(E)$ soit bornée ce qui entraîne γ absolument continue. Suit la construction d'un système orthonormé et complet $h_i(x)$ de fonctions φ. Si P est continue, on obtient la valeur de $P(f)$ au moyen du développement de f suivant les h_i. Cette valeur est essentiellement égale à $(D)\int fd\gamma$.

§5. Soit $\alpha[f]$ un opérateur linéaire continu en moyenne d'ordre $r \geqq 1$, $\varphi_I(x)$ la fonction caractéristique de l'intervalle I. Alors les sommes définissant $(D)\int fd\alpha[\varphi_I(x)]$ convergent en moyenne d'ordre r vers $\alpha[f(x)]$.

En conclusion, l'auteur envisage des extensions au cas de n ou d'une infinité de dimensions, au cas d'une mesure de base autre que celle de Lebesgue, et au cas d'une norme autre que la moyenne d'ordre r. *R. de Possel* (Alger).

Jacobsthal, Ernst. Zur Theorie der reellen Funktionen. Norske Vid. Selsk. Forh., Trondheim 23, 83–86 (1951).

Suppose that $f(x)$ is real-valued on (a, b) and has the Darboux property that on every sub-interval (c, d) it assumes every value between $f(c)$ and $f(d)$. The writer points out that if $f(x)$ is schlicht (assumes each of its values once only) then $f(x)$ is necessarily monotonic and continuous.

I. Halperin (Kingston, Ont.).

Motchane, Léon. Sur les critères de conservation de classe et les familles de fonctions fermées au sens de la convergence simple. C. R. Acad. Sci. Paris **231**, 1206–1208 (1950).

$g(x)$ désigne une fonction de la première classe de Baire, définie sur $(0, 1)$, P un ensemble parfait quelconque, E_P l'ensemble des points de P où $g(x)$ est continue relativement à P. Les oscillations inférieure $\underline{\omega}^e(P; x_0)$ et supérieure $\overline{\omega}^e(P; x_0)$ en un point x_0 de P sont définies respectivement comme la limite inférieure et la limite supérieure de $|g(x)-g(x_0)|$ quand $x\to x_0$ sur E_P. La fonction $g(x)$ est quasi-continue si $\underline{\omega}^e(P; x)=0$. L'oscillation classique d'une fonction f en un point x_0 de $(0, 1)$ est représentée par $\overline{\omega}^r(f; x_0)$. La fonction $h(x)$ désigne la limite d'une suite (ponctuellement) convergente $g_n(x)$ de fonctions de première classe. Voici des spécimens de critères de conservation de classe: Si les g_n sont quasi-continues, h est de première classe. Pour que h soit de première classe, il faut et il suffit que sur tout P et pour tout $\epsilon>0$, l'ensemble $E_\epsilon[\limsup \underline{\omega}_n^e(P; x) \geqq \epsilon]$ soit non dense sur P. Si $S(x)=\limsup \overline{\omega}^r(g_n; x)$ est de première classe et nulle en ses points de continuité, h est de première classe. Une famille $\mathfrak{F}(g)$ de fonctions g est dite équi-quasi-continue si pour tout $\epsilon>0$ et pour tout voisinage V de tout point x_0, il existe un sous-intervalle i indépendant de g et inclus dans V tel que $|g(x_0)-g(x)|<\epsilon$ pour $x\epsilon i$, $g\epsilon\mathfrak{F}$. De toute famille infinie de fonctions équi-quasi-continues bornée en tout point x de $(0, 1)$ on peut extraire une suite convergente. La limite de cette suite est quasi-continue. Pour que la limite d'une suite de fonctions de première classe soit quasi-continue, il faut et il suffit que la suite soit équi-quasi-continue sauf, en chaque point, au plus pour un nombre fini de termes. Ces théorèmes sont valables pour des familles de fonctions équi-approximativement continues en chaque point. Pour les familles des types précités, la convergence sur un ensemble $\{a_n\}$ partout dense sur $(0, 1)$ entraîne la convergence partout des suites de fonctions. Cette propriété remarquable permet d'exprimer la convergence simple (ou ponctuelle) dans ces familles au moyen de la métrique (suggérée par G. Choquet) $\delta(g', g'')=\sum \delta_i(g', g'')/2^i$ où $\delta_i(g', g'')$ représente la distance réduite (suivant Hahn) de $g'(a_i)$ et $g''(a_i)$. Avec cette métrique l'espace des fonctions équi-quasi-continues (ou équi-approximativement continues) et bornées en un point au moins est pré-compact. Une publication est annoncée contenant une caractérisation de tous les espaces compacts au sens de la convergence simple. [Remarque: La fonction g considérée au début est supposée dans la note seulement mesurable B. Le référent ne comprend les définitions de l'auteur qu'en la supposant de première classe.] *C. Pauc* (le Cap).

Gál, I. S. Sur la majoration des suites de fonctions. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 243–251 (1951).

General ideas developped in a joint paper with J. F. Koksma [same Proc. **53**, 638–653 = Indagationes Math. **12**, 192–207 (1950); these Rev. **12**, 86] are applied here to some concrete cases. Suppose that a sequence of functions $F(M, N; x) \geqq 0$ $(M, N=0, 1, \cdots)$ satisfies the conditions

$$F(M, 0; x)=0,$$

$$F(M, N; x) \leqq F(M, N', x)+F(M+N', N-N'; x) \qquad (0 \leqq N' \leqq N),$$

$$\int_0^1 F(M, N; x)^{2p} dx = O\left(\left(\frac{\gamma p N}{e}\right)^p e^{(3p/N)+k\log p}\right),$$

where $\gamma>0$ and $k\geqq 0$ are constants, $M=0, 1, \cdots, N=0, 1, \cdots$ and $p=1, 2, \cdots \leqq \log\log (M+N)$. Then

$$\limsup_{N\to\infty} F(0, N; x)/(\gamma N \log\log N)^{\frac{1}{2}} \leqq 1,$$

almost everywhere in $0\leqq x\leqq 1$. In particular, if $f_1(x)$, $f_2(x), \cdots$ are Rademacher functions, then

$$\limsup_{N\to\infty} \sum_{\nu=1}^{N}\left(1-\frac{\nu-1}{N}\right)f_\nu(x)/(\tfrac{2}{3}N\log\log N)^{\frac{1}{2}} \leqq 1$$

almost everywhere. *A. Zygmund* (Chicago, Ill.).

Cecconi, Jaurès. Un esempio nella teoria delle trasformazioni piane. Boll. Un. Mat. Ital. (3) 6, 18–21 (1951).

In view of the number of basic definitions to be employed, a detailed statement of all of them is not feasible here. For concepts used without explanation in the sequel, the reader may consult the reviewer's book on Length and Area [Amer. Math. Soc. Colloq. Publ., vol. 30, New York, 1948, chapter IV; these Rev. **9**, 505]. Let $T: x=x(u, v), y=y(u, v), (u, v)\epsilon Q$: $0\leqq u\leqq 1$, $0\leqq v\leqq 1$, be a continuous mapping from the unit square Q into the (x, y)-plane. Let C be the parametrized closed continuous curve which corresponds to the perimeter of Q under T. Each point (x, y) has then a topological index $o(x, y)$ with respect to C, where $o(x, y)=0$ if (x, y) lies on C. Also, each point (x, y) has a signed multiplicity $n(x, y)$ which depends upon the behavior of T in all of Q. It is known that if T is of bounded variation, then $o(x, y)=n(x, y)$ almost everywhere on the complement of C. On C itself, $o(x, y)=0$ by definition, while no general statement can be made concerning $n(x, y)$ for (x, y) on C. The example constructed by the author shows that $o(x, y)$ and $n(x, y)$ may differ from each other on a set of positive measure in case the point-set occupied by C is of positive measure. Actually, the author constructs an example of two continuous mappings T_1, T_2 with the same C such that the corresponding functions $n_1(x, y)$, $n_2(x, y)$ differ from each other on a set of positive measure. *T. Radó* (Columbus, Ohio).

Theory of Functions of Complex Variables

Goodman, A. W. Typically-real functions with assigned zeros. Proc. Amer. Math. Soc. **2**, 349–357 (1951).

Let $T(p)$ denote the class of functions $f(z)=\sum_{n=0}^{\infty} b_n z^n$ which are typically-real of order p with respect to the unit circle. This means that the coefficients b_n are all real and that either (1) $f(z)$ is regular in $|z|\leqq 1$ and the imaginary part of $f(z)$ changes sign $2p$ times on $|z|=1$, or (2) $f(z)$ is regular in $|z|<1$ and for each r in a range $\rho<r<1$, the imaginary part of $f(z)$ changes sign $2p$ times on $|z|=r$. The class $T(1)$ was first introduced by W. Rogosinski [Math. Z. **35**, 93–121 (1932)] and for $p>1$ by M. S. Robertson [Ann. of Math. (2) **38**, 770–783 (1937); also Duke Math. J. **5**, 512–519 (1939); these Rev. **1**, 9]. The sharp upper bound for $|b_n|$ in terms of $|b_1|, \cdots, |b_p|$ was recently obtained by A. W. Goodman and M. S. Robertson [Trans. Amer. Math. Soc. **70**, 127–136 (1951); these Rev. **12**, 691].

In this paper the author obtains sharp bounds for the coefficients $|b_n|$ when the zeros of $f(z)$ within the unit circle are assigned. Let $f(z)$ belong to $T(p)$ with a zero at the origin of order q and exactly s zeros β_j for which $0<|\beta_j|<1$. Let $b_q=1$ and $m=[\frac{1}{2}(p-q-s+1)]$. Then $|b_n|\leqq B_n$, $n>q$, where B_n are defined by

$$F(z)=\frac{z^q}{(1-z)^{2q+2s}}\left(\frac{1+z}{1-z}\right)^{2m}\prod_{j=1}^{s}\left(1+\frac{z}{|\beta_j|}\right)(1+z|\beta_j|) = \sum_{n=q}^{\infty} B_n Z^n.$$

When $p-q-s$ is odd or zero, $F(z)\epsilon T(p)$ and $|b_n|=B_n$ for $f(z)\equiv F(z)$. The author suggests that a similar set of inequalities may hold for the coefficient of multivalent functions with assigned zeros within the unit circle.

M. S. Robertson (New Brunswick, N. J.).

Eggleston, H. G. A Tauberian lemma. Proc. London Math. Soc. (3) **1**, 28–45 (1951).

Using elementary methods (Taylor's expansion and Cauchy's inequality) the author gives theorems describing the behaviour of a bounded analytic function (or that of its derivatives) at a boundary point z_0 of its domain of definition D. If L is an arc in D with an endpoint z_0, if $f(z)\to 0$ for $z\to z_0$ along L, and if $d(z)$ denotes the distance from z to the boundary of D, then (a) $f^{(p)}(z)d(z)^p\to 0$, $p=1, 2, \cdots$, along L; (b) for any $0<\chi<1$, $f(z_1)\to 0$ as $z_1\to z_0$ in such a way that there are points $z\epsilon L$ with $|z_1-z|<\chi d(z)$. In other theorems D is a strip or an angle, L a straight line and $f(z)$ approaches 0 when $z\to z_0$ along a set on L of "positive h-density"; the conclusion is then changed accordingly. There are applications to almost periodic functions which are analytic in a strip. Finally, theorem (a) is shown to lead in a natural way to a new proof of Littlewood's Tauberian theorem for general Dirichlet series.

G. G. Lorentz (Toronto, Ont.).

Mori, Akira. Valiron's theorem on Picard's curves. Kōdai Math. Sem. Rep. **1950**, 101–103 (1950).

The theorem of Valiron referred to in the title is of the following nature: Let $f(z)$ be meromorphic for $|z|\geqq 1$, and consider the region $\Delta(k, \delta, \epsilon)$ swept out by the discs $|z-z_0|<\epsilon|z_0|$ when z_0 traces the spiral $z_0=e^{i\delta}t^{1+ik}$; for all δ, ϵ and almost all real k the function $f(z)$ takes all values in $\Delta(k, \delta, \epsilon)$ with at most two exceptions. The author generalizes the theorem, proving that one out of three disjoint discs will be covered by infinitely many simply connected islands, and one out of five discs will be simply covered. For this result, suggested by the reviewer's generalization of Picard's theorem, it is necessary to assume that $f(z)$ is not exceptional in the sense of Julia (for $\sigma_n\to\infty$ the family $\{f(\sigma_n z)\}$ shall be normal), but it is shown that a stronger form of Valiron's theorem follows by the same method regardless of the assumption. *L. Ahlfors.*

Majstrenko, P. On a theorem of Poincaré and Volterra. Proc. London Math. Soc. (2) **53**, 57–64 (1951).

The author gives a new proof of the theorem that if $f(z)$ is an analytic function the set of distinct values $f(p)$ is denumerable for each point p in the complex plane. The standard proof involves the construction of an explicitly chosen denumerable set of Taylor series from which all values of $f(z)$ at all points can be determined. By means of a certain geometrical technique the author avoids this explicit construction. He also sketches an abstract treatment of his method, to be developed in a later paper.

G. Piranian (Ann Arbor, Mich.).

Carleson, Lennart. On null-sets for continuous analytic functions. Ark. Mat. 1, 311–318 (1951).

Let E be a compact set in the z-plane, Ω its complement and let Γ be the class of functions $f(z)$ which are analytic possibly multivalued in Ω and have a certain property P. Then E is said to be a nullset for P if Γ consists entirely of constants. Let $L_\alpha(E)$, $C_\alpha(E)$ denote Hausdorff measure and capacity of order α, $0<\alpha<2$. The following are among the results proved by methods similar to the author's thesis [Upsala, 1950; these Rev. **11**, 427]: E is a nullset for the class Γ_1 of functions belonging to Lip α and having single-valued real part if and only if $L_\alpha(E)=0$. If Γ_2 is the subclass of functions whose imaginary part is also single-valued it is sufficient for E to be a nullset that $L_{1+\alpha}(E)=0$ and necessary that $C_{1+\alpha}(E)=0$. If Γ_3 is the class of single-valued functions with uniformly continuous derivative outside E, then E is a nullset if and only if it has no inner points.

W. K. Hayman (Exeter).

Wille, R. J. On the number of doublepoints of analytic curves. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 178–183 (1951).

Soit $f(z)$ une fonction méromorphe sur la circonférence C, et soit $I=f(C)$ l'image de C. Un point A de I est dit point double si les deux conditions suivantes sont satisfaites: (1) Il existe plus d'un point $z \epsilon C$ correspondant à A; (2) ces valeurs z n'admettent pas de voisinages disjoints sur C avec images identiques. Théorème: Le nombre de points doubles de I est fini.

S. Mandelbrojt (Houston, Tex.).

Fridman, G. A. Determination of the character of an isolated singularity of an analytic function from the moduli of the coefficients of two of its power series expansions. Doklady Akad. Nauk SSSR (N.S.) **75**, 341–344 (1950). (Russian)

Un point singulier isolé z_0 d'une fonction analytique $\psi(z)$ uniforme autour de z_0 appartient à la classe $[\rho, \mu]$ si la fonction entière $h(z)$, telle que la fonction $h(1/(z-z_0))-\psi(z)$ est régulière en z_0, est de l'ordre $<\rho$ et du type $\leqq\mu$. En généralisant un théorème classique de Pólya, l'auteur démontre les théorèmes suivants: (1) Si la fonction analytique uniforme $\psi(z)$ admet toutes ses singularités sur le cercle de rayon un, et si $\psi(z)=\sum_0^\infty a_n z^n$, $|z|<1$, $\psi(z)=\sum_1^\infty C_n z^{-n}$, $|z|>1$, avec $|a_n|=O(e^{\kappa n^\sigma})$, $|C_n|=O(e^{\kappa n^\sigma})$, $0<\sigma<1$, alors toute singularité isolée est de la classe $[\rho, \mu]$, où $\rho=\sigma/(1-\sigma)$, $\mu=[\kappa\sigma \sec \frac{1}{2}\pi\sigma]^{1+\rho}/\rho$. (2) Si $\psi(z)$ est analytique et uniforme, si toutes ses singularités qui sont dans l'anneau $0\leqq r'<|z|<r''\leqq\infty$ sont situées sur la circonférence $|z|=r$, $r'<r<r''$, et si $\psi(z)=\sum_{-\infty}^\infty C_n' z^n$, $r'<|z|<r$; $\psi(z)=\sum_{-\infty}^\infty C_n'' z^n$, $r<|z|<r''$, avec $|C_n'|r^n=O(e^{\kappa n^\sigma})$, $n\to\infty$; $|C_{-n}''|r^{-n}=O(e^{-\kappa n^\sigma})$, $n\to\infty$, alors toute singularité isolée de ψ sur $|z|=r$ est de la classe $[\rho, \mu]$. Ces théorèmes peuvent être généralisés, comme l'indique l'auteur, au cas où κn^σ est remplacé par $\sigma(n)$, où $\sigma(n)$ est une fonction non décroissante telle que $\int^\infty[\sigma(x)/x^2]dx<\infty$ ainsi qu'aux séries de Dirichlet.

S. Mandelbrojt (Houston, Tex.).

Schweitzer, M. The partial sums of second order of the geometric series. Duke Math. J. **18**, 527–533 (1951).

Two examples are given which show the importance of a study of the positivity of trigonometric polynomials as the variable ranges over a fixed interval which may be only a part of the period. In particular, if

$$S_n^{(2)}(z)=\sum_{\nu=0}^{n}\binom{n+2-\nu}{2}z^\nu$$

denotes the sum by arithmetic means of order two of the geometric series, it was proved by Szegö [Duke Math. J. **8**, 559–564 (1941); these Rev. **3**, 76] that the real part of $S_n^{(2)}(e^{i\phi})$ is decreasing for $0\leqq\phi\leqq\gamma$, $\sin^2\frac{1}{2}\gamma=0.7$, $\frac{1}{2}\pi<\gamma<\pi$. The author shows that γ can be replaced by $\frac{2}{3}\pi$, and by no larger number. This extends an analogous result of Fejér [Z. Angew. Math. Mech. **13**, 80–88 (1933)] for large n. The introduction and publication of this posthumous note was taken care of by L. Fejér and G. Szegö.

M. S. Robertson (New Brunswick, N. J.).

Seleznev, A. I. On universal power series. Mat. Sbornik N. S. **28**(70), 453–460 (1951). (Russian)

Désignons par S l'ensemble des fonctions f dont chacune est définie sur un ensemble correspondant E_f par la somme d'une série de polynomes. Soit E_f^* l'ensemble de régularité de cette série par rapport à E_f ($z \epsilon E_f^*$, si $z \epsilon E_f$ et si la série de polynomes converge uniformément sur $\Delta_z \cap E_f$, Δ_z étant un certain voisinage de z). Une série (1) $\sum_1^\infty \alpha_\nu z^\nu$ est dite universelle si à chaque $f \epsilon S$ et à chaque arc de Jordan γ joignant les points 0 et ∞ correspond une suite de polynomes-sections de (1) tendant vers $f(z)-f(0)$ sur E_f, dont chaque point de E_f^* n'appartenant pas à γ est un point de régularité. On démontre les théorèmes suivants: (1) Toute série $\sum_1^\infty a_\nu z^\nu$ peut être décomposée en deux séries universelles $\sum_1^\infty \alpha_\nu z^\nu$, $\sum_1^\infty (a_\nu-\alpha_\nu)z^\nu$ telles que pour chaque ν un des coefficients α_ν, $a_\nu-\alpha_\nu$ appartienne à un ensemble donné R, dénombrable, partout dense. (2) Soit f une fonction mesurable sur un ensemble mesurable, borné E (la partie réelle et la partie imaginaire de f sont mesurables sur E). Il existe une suite d'ensembles fermés $\{F_n\}$ tels que $F_k \subset E$, $F_k \subset F_{k+1}$ ($k\geqq1$), $\mu(\bigcup F_k)=\mu E$ et tels qu'une suite de polynomes-sections de (1) converge sur $\bigcup F_k$ et uniformément sur chaque F_k. Les ensembles S ont été introduits par Lavrentieff [Sur les fonctions d'une variable complexe représentables par des séries de polynomes, Actualités Sci. Ind. no. **441**, Hermann, Paris, 1936]. Les théorèmes (1), (2) sont analogues aux théorèmes de Menchoff concernant les séries universelles trigonométriques [C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 79–82 (1945); Mat. Sbornik N.S. **20**(62), 197–238 (1947); ces Rev. **7**, 435; **8**, 577]. Des théorèmes analogues aux théorèmes (1), (2) sont démontrés pour la variable réelle.

S. Mandelbrojt (Houston, Tex.).

Mergelyan, S. N. On the representation of functions by series of polynomials on closed sets. Doklady Akad. Nauk SSSR (N.S.) **78**, 405–408 (1951). (Russian)

The author states and proves the following theorem. Let E be a bounded closed point-set in the plane, which does not separate the plane. Let $f(z)$ be a continuous function defined on E which is holomorphic at every interior point of E. Then there exists a sequence of polynomials $\{P_n(z)\}$ such that $P_n(z)\to f(z)$ uniformly on E. This theorem contains as special cases the theorems of Lavrentieff [Sur les fonctions d'une variable complexe . . . , Actualités Sci. Ind. no. 441, Hermann, Paris, 1936] in which E is nowhere dense, Walsh [Math. Ann. **96**, 430–436 (1926)] in which E is the closure of a Jordan domain, and Keldysh [Mat. Sbornik. N.S. **16**(58), 249–258 (1945); these Rev. **7**, 285] in which E is the closure of its interior. The short ingenious proof uses the method by which the author [same Doklady **77**, 565–568 (1951); these Rev. **12**, 814] recently proved Lavrentieff's theorem.

L. Bers (Los Angeles, Calif.).

Krasnoščekova, T. I. A theorem on series of polynomials. Doklady Akad. Nauk SSSR (N.S.) 77, 787–789 (1951). (Russian)

Let G be a simply connected region in the z-plane, bounded by a Jordan curve C, and of transfinite diameter r. Let $\varphi(z)$ be that function which maps the exterior of G conformally upon the region $|w|>r$ and which has the form $\varphi(z)=z+\alpha_0+\alpha_1/z+\alpha_2/z^2+\cdots$ in the neighborhood of the point ∞; and for $R>r$, let C_R be the image under $\varphi(z)$ of the circle $|w|=R$. Let $\{P_n(z)\}$ be any sequence of polynomials of the form $z^n+\cdots+a_0^{(n)}$ whose elements satisfy the requirement that for every $R>r$ and for every $\epsilon>0$ an inequality $|P_n(z)|<M(R+\epsilon)^n$ holds on C_R, M being a constant that depends on ϵ and R, but not on n. Finally, let $\{c_n\}$ be any sequence of numbers with the property that $\lim\sup|c_n|^{1/n}=1/\rho$ $(\rho>r)$. The author proves that there exists a sequence $\{\epsilon_n\}$ $(\epsilon_n=\pm1)$ such that the curve C_ρ is a natural boundary for the function $\sum\epsilon_nc_nP_n(z)$. The proof follows the pattern of Hurwitz' proof for the corresponding theorem on Taylor series [A. Hurwitz and G. Polya, Acta Math. 40, 179–183 (1916)]. It depends on Al'per's result on the overconvergence of series of polynomials [Doklady Akad. Nauk SSSR (N.S.) 59, 625–627 (1948); these Rev. 9, 422], and on the following lemma: If $\{Q_n(z)\}$ $(n=n_1, n_2, \cdots)$ is a sequence of polynomials, converging uniformly in every closed region interior to G, and if $Q_n(z)$ is of degree n and has the leading coefficient $c_n^{(n)}$, then $\lim\sup|c_n^{(n)}|^{1/n}\leqq1/r$.

G. Piranian (Ann Arbor, Mich.).

Berghuis, J. Computation of the coefficients of the asymptotic development of the function

$$\Omega(z)=\sum_{n=0}^{\infty}\frac{z^n}{\prod_{i=1}^{p}\Gamma(\alpha_in+\beta_i)}.$$

Math. Centrum Amsterdam. Rapport **ZW 1951-007**, 12 pp. (1951). (Dutch)

The same problem was considered by H. K. Hughes [Bull. Amer. Math. Soc. 51, 456–461 (1945); these Rev. 6, 263]. The author's method is different, and makes use of results of Kemperman [same report series, Rapport **ZW 1950-018**; these Rev. 12, 600]; the author is able to give the first few coefficients in the asymptotic expansion explicitly.

R. P. Boas, Jr. (Evanston, Ill.).

***Orton, William Rolen, Jr. Representation of functions of a complex variable and related integral equations.** Abstract of a Thesis, University of Illinois, (1951). i+3 pp.

"The present work generalizes a representation theorem of V. Fedoroff [Mat. Sbornik N.S. 2(44), 521–541 (1937)] by extending the class of functions representable by

$$f(z)=\int_D[u(t)/(t-z)]ds+A(z),$$

z in D."

Extract.

***Sergeev, N. S. Issledovanie odnogo klassa transcendentnyh funkcii, opredelyaemyh obobščennym uravneniem Rimana. [Investigation of a Class of Transcendental Functions Defined by a Generalized Riemann Equation].** Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1949. 154 pp.

I. The generalized functions of Riemann $\zeta_p(s)$, $\eta_p(s)$ and their fundamental properties. II. Generalized Bernoulli polynomials of the first kind. III. On summation formulas for sums of the form $\sum_{j=1}^{\infty}\{(p^2+\lambda_j^2)/[p(p+1/\pi)+\lambda_j^2]\}f(\lambda_j)$. IV. On gammamorphic functions $\Gamma_{1,p}(x)$ of the first kind. V. Generalized Bernoulli polynomials of the second kind. VI. On summation formulas for sums of the form $\sum_{k=1}^{\infty}\int_0^\infty e^{-kx}[(2\pi p-x)/(2\pi p+x)]^k f(x)dx$. VII. The generalized Poisson formula. VIII. On gammamorphic functions $\Gamma_{2,p}(x)$ of the second kind. IX. The generalized formula of Ramanujan.

Table of contents.

***Neville, Eric Harold. Jacobian Elliptic Functions.** 2d ed. Oxford, at the Clarendon Press, 1951. xvi+345 pp. $7.00.

Except for the addition of a chapter on "Integrals of the third kind", this is a photographic reprint of the first edition [1944; cf. these Rev. 7, 53] with corrections in the text.

Springer, G. The coefficient problem for schlicht mappings of the exterior of the unit circle. Trans. Amer. Math. Soc. 70, 421–450 (1951).

Let T be the class of functions of the form

$$w=f(z)=z+a_1/z+a_2/z^2+\cdots,$$

which are schlicht in $|z|>1$ and let S be the class of functions $z=\phi(w)=w+b_1/w+b_2/w^2+\cdots$, inverse to $f(z)$. Clearly $a_1=-b_1$, $a_2=-b_2$. The inequality $|a_1|\leqq1$ is a classic consequence of the area principle. Golusin [Rec. Math. [Mat. Sbornik] N.S. 3(45), 321–330 (1938)] and Schiffer [Bull. Soc. Math. France 66, 48–55 (1938)] proved $|a_2|\leqq\frac{2}{3}$, with equality only for $f(z)=z(1+e^{i\alpha}/z^3)^{\frac{2}{3}}$. The author now proves $|b_3|\leqq1$ with equality for $f(z)=z+e^{i\alpha}/z$ only. He shows further that the functions maximising $|b_n|$ satisfy a hyperelliptic differential equation in z, w and that this equation is satisfied if $w=f(z)=z(1+z^{-q})^{2/q}$ is the function which maps the exterior of the unit circle onto the domain bounded by q symmetric radial slits meeting at the origin, where q is the highest prime divisor of $n+1$. This leads to the conjecture (true for $n=1, 2, 3$) that these functions do indeed give the extreme value for $|b_n|$. The results are based on a variational formula for the mapping function formally due to Julia [Ann. Sci. École. Norm. Sup. (3) 39, 1–28 (1922)], whose rigorous justification in this case constitutes one of the main technical difficulties of the proof.

W. K. Hayman (Exeter).

Fejér, L., and Szegö, G. Special conformal mappings. Duke Math. J. 18, 535–548 (1951).

The paper is concerned with special conformal mappings $w=f(z)$ of the circle $|z|\leqq1$. A compilation is made of 22 elementary formulas, useful in studying the maps of $|z|=$constant or arc $z=$constant. For the class of mappings $w=f(z)=u(r,\theta)+iv(r,\theta)$ for which (a) $f(z)$ is regular in $|z|\leqq1$, (b) $f(z)$ is real for z real, (c) $\partial u(r,\theta)/\partial\theta<0$ for $r=1$ and $0<\theta<\pi$, (d) $v(r,\theta)>0$ for $r=1$ and $0<\theta<\pi$, which was introduced first by Fejér [J. London Math. Soc. 8, 53–62 (1933)], it is shown that (a), (b), and (c) imply (d) and that $\partial v/\partial r>0$ for $0<\theta<\pi$.

The authors discuss the Cesàro sums of order k of the geometric series:

$$S_n^{(k)}(z)=\sum_{\nu=0}^{n}\binom{n+k-\nu}{k}z^\nu$$

which are of importance in the study of power series whose coefficients are monotonic of order $k+1$. New proofs of the theorems of Egerváry [Math. Z. 42, 221–230 (1937)] that $w=S_n^{(1)}(z)$ provides a star-like mapping with respect to the point $S_n^{(1)}(1)$ and that $w=S_n^{(3)}(z)$ is convex for $|z|\leqq1$ are

given. It is observed that $w=S_n^{(k)}(z)$ for $k\geqq 3$ are convex in the vertical direction. The question is raised (but not settled) whether $S_n^{(k)}(z)$ are also convex for $k\geqq 3$, $|z|\leqq 1$. An example is given of a power series $f(z)$, convergent for $|z|<1$, having coefficients which form a totally monotonic sequence, and for which $f(z)$ is not convex for $|z|<1$, even though $f(z)$ is convex in the vertical direction.

M. S. Robertson (New Brunswick, N. J.).

Nehari, Zeev. Sur la représentation conforme de deux domaines complémentaires. C. R. Acad. Sci. Paris **232**, 1532–1534 (1951).

In this note the author gives a necessary and sufficient condition in order that $f(z)=\sum_1^\infty a_m z^m$ and $g(z)=\sum_{-1}^{-\infty} b_m z^m$ ($g(z)\neq 0$) will map the unit circle conformally onto the interior and the exterior, respectively, of the same analytic Jordan curve. The condition is that

$$\sum_{m=1}^{\infty} m[|a_m^{(n)}|^2+|b_m^{(n)}|^2]=-\sum_{m=-\infty}^{-1} m|b_m^{(n)}|^2 \quad (n=1, 2, \cdots),$$

where $a_m^{(n)}$ and $b_m^{(n)}$ are the coefficients of the Taylor and the Laurent expansions of $P_n[f(z)]$ and $P_n[g(z)]$, $P_n(w)$ being an arbitrary polynomial of degree n. The paper suffers from a number of printing mistakes. *S. Agmon.*

Fourès, Léonce. Sur les surfaces de recouvrement régulièrement ramifiées. C. R. Acad. Sci. Paris **232**, 467–469 (1951).

Given a closed Riemann surface R and points a_n on R, a covering surface R^* of R is said to be regularly ramified if R^* has branch-points of order p_n at all points corresponding to a_n and if elsewhere the behavior of R^* is the same as that of R at the corresponding points. A number of results involving this concept are stated and the relation between this definition and that of a regularly ramified graph which appeared in an earlier paper [same C. R. **230**, 353–355 (1950); these Rev. **11**, 590] is discussed. *Z. Nehari.*

Levin, B. Ya. Some extremal properties of entire functions of several variables. Doklady Akad. Nauk SSSR (N.S.) **78**, 861–864 (1951). (Russian)

The author extends some of his earlier results on entire functions of exponential type [same Doklady (N.S.) **65**, 605–608 (1949); Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 45–84 (1950); these Rev. **11**, 23, 510] to functions of two complex variables, thus obtaining in particular a "Bernstein inequality" for partial derivatives. He points out that similar work by Meĭman [same Doklady (N.S.) **71**, 609–612 (1950); these Rev. **11**, 509] does not cover derivatives of higher order, while his own work does. *R. P. Boas, Jr.*

Levin, B. Ya. On a class of entire functions. Doklady Akad. Nauk SSSR (N.S.) **78**, 1085–1088 (1951). (Russian)

Define $\overline{HB}$ as the class of entire functions $\omega(z)$ which in the lower half plane have no roots and satisfy $|\omega(z)|\geqq|\omega(\bar z)|$; P^* is the subclass of $\overline{HB}$ of the form $\exp(-\gamma z^2)\omega_1(z)$, where $\gamma\geqq 0$ and $\omega_1(z)$ is of the first order at most; this is the same as the class of limits, uniformly in every bounded domain, of polynomials with no roots in the lower half plane (consequence of the Laguerre-Pólya theorem). The author gives several theorems about P^*. If $\omega(z)$ is entire and $[e^{i\tau}\omega(z)]'\epsilon\overline{HB}$ for every real τ, then $\omega(z)\epsilon P^*$. If $\omega(z)$ is entire it belongs to P^* if and only if $\omega(z-ih)\epsilon\overline{HB}$ for every positive h. If $|f(z)|$ is entire, $f(z)-\nu\omega(z)$ belongs to P^* for every ν, $|\nu|\geqq 1$, if and only if $\omega(z)\epsilon P^*$ and majorizes $f(z)$, i.e. $|\omega(z)|\geqq f(z)$ and $|\omega(z)|\geqq|f(\bar z)|$ for $\mathfrak{J}(z)\leqq 0$. A $\mathfrak{B}^*$-operator is an additive homogeneous operator defined on the linear manifold generated by P^* and leaving P^* invariant. If it is moreover continuous in the topology of uniform convergence on every bounded domain and is applied (with respect to one variable) to a function $\omega(z, u)$ of two complex variables which belongs to $\overline{HB}$ in the sense of the paper reviewed above, it preserves the property (A): $\omega(z, u)\epsilon\overline{HB}$, $\omega(z, u)\epsilon P_z^*$ for $\mathfrak{J}(u)\leqq 0$, $\omega(z, u)\epsilon P_u^*$ for $\mathfrak{J}(z)\leqq 0$. Using this the author can extend the Laguerre-Pólya theorem to two variables as follows. Condition (A) is necessary and sufficient for the entire function $\omega(z, u)$ to be the limit (in the topology mentioned above) of polynomials which have no zeros for $\mathfrak{J}(u)<0$, $\mathfrak{J}(z)<0$, i.e. polynomials belonging to $\overline{HB}$.

R. P. Boas, Jr. (Evanston, Ill.).

Garabedian, P. R. A new formalism for functions of several complex variables. J. Analyse Math. **1**, 59–80 (1951).

The author investigates, in a frankly heuristic manner, certain aspects of the theory of functions of several complex variables. His main aim is to find an analogue to the relation, established by Bergman and Schiffer [Duke Math. J. **14**, 609–638 (1947); these Rev. **9**, 187], between the kernel function and the Green's and Neumann's functions of a partial differential equation of elliptic type. The procedure consists in formally regarding the extremal problem within the class of solutions of a given partial differential equation, which defines the kernel function, as a problem in the ordinary calculus of variations with the differential equation as a side condition. This leads to the existence of Lagrange multipliers λ which vanish on the boundary and have certain point singularities, and thus show the characteristic behavior of Green's functions. In the case of functions of several complex variables, the λ's are shown to be solutions of a partial differential equation generalizing the Laplace equation, and the kernel function can be expressed in terms of the λ's in a simple fashion. The same procedure is also applied to a problem connected with the pseudo-conformal mapping of one four-dimensional region onto another by means of a pair of functions of two complex variables. While none of the results are rigorously established, the formalism developed gives considerable insight into the nature of the problems and it may well serve as a guide for a future rigorous treatment of the subject. *Z. Nehari.*

✶Staub, Alfred. Integralsätze hyperkomplexer, regulärer Funktionen von 2n reellen Variablen. Thesis, University of Zürich, 1946. 43 pp.

Der Autor betrachtet analytische Funktionen

$$u=\sum_{k=0}^{n-1} i_k[u_{2k}+iu_{2k+1}]$$

in einem Clifford'schen Produktsystem

$$\mathfrak{P}_{2n}=[1, i][1, i_1, i_2, \cdots, i_{n-1}]$$

der Dimension $2n$. Unter diesen findet man als Spezialfall die komplex-analytischen Funktionen $u=\sum_{k=0}^{n-1} i_k w_k$, wo die w_k im gewöhnlichen Sinn analytische Funktionen von n komplexen Variablen sind. Für die analytischen Funktionen u gilt der verallgemeinerte Cauchy'sche Integralsatz [Fueter, Comment. Math. Helv. **14**, 394–400 (1942); diese Rev. **4**, 139] $\int_{(R)} u dZ=0$, wobei u in G analytisch ist und R den Rand eines innern Gebietes $G'\subset G$ bezeichnet. Der Verfasser

benützt nun diesen Satz um für die geschlossenen Flächen der Dimensionen n bis $2n-1$ folgenden Integralsatz zu beweisen:

$$\int_{(O^{n+h})}\left\{\sum_{\beta_1=1}^{n-1} u^{(\alpha_{\beta_1})} Z_{\alpha_{\beta_2}\cdots\alpha_{\beta_{n-h}}}\right\}\prod_{\gamma=1}^{n+1} dt_\gamma=0, \quad h=0, 1, \cdots, n-1.$$

O^{n+h} ist eine im endlichen R^{2n} gelegene, orientierte, geschlossene, sich nirgends durchdringende und stückweise analytische Fläche von der Dimension $n+h$. u ist in einer $2n$-dimensionalen Umgebung der Fläche O^{n+h} eine analytische Funktion. $\alpha_1, \cdots, \alpha_{n-h}$ ist eine beliebige, aber feste Kombination zur $(n-h)$ten Klasse der Zahlen $0, 1, 2, \cdots, 2n-1$. $\alpha_{\beta_1}, \cdots, \alpha_{\beta_{n-h}}$ bedeutet jeweils die natürliche Anordnung aus der Zahlenfolge $\alpha_1, \cdots, \alpha_{n-h}$, worin die Zahl α_{β_1} fehlt. Da dieser Satz natürlich auch für komplex-analytische Funktionen gilt, und in den letztern keine Beziehungen zwischen den komplexen Komponenten w_k bestehen, kann er für eine einzige solche Komponente $u=w_0=f(z_0, z_1, \cdots, z_{n-1})$ angewendet werden. Der Autor zeigt, dass er sich in diesem Fall auf den Integralsatz von Martinelli [Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **9**, 269–283 (1938)] reduziert:

$$\int_{(O^{n+h})} f(z_0, z_1, \cdots, z_{n-1})dz_0dz_1\cdots dz_{n-1}d\bar{z}_{\epsilon_1}d\bar{z}_{\epsilon_2}\cdots d\bar{z}_{\epsilon_h},$$

wo $\epsilon_1, \epsilon_2, \cdots, \epsilon_h$ eine beliebige, feste Kombination von h Zahlen der Folge $0, 1, \cdots, n-1$ bedeutet. Damit hat er bewiesen, dass der Integralsatz von Martinelli und mit diesem die Integralsätze von Poincaré ($h=0$) und Wirtinger ($h=n-1$) im verallgemeinerten Cauchy'schen Integralsatz für analytische Funktionen des Clifford'schen Produktsystems $\mathfrak{P}_{2n}$ enthalten sind. *H. G. Haefeli.*

Myrberg, P. J. Beispiele von automorphen Funktionen zweier Variablen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **89**, 16 pp. (1951).

In a previous paper [same Ann. no. **53** (1948); these Rev. **10**, 525] the author has considered the class of (automorphic) functions of two complex variables which are invariant under certain groups of birational (Cremona) transformations. In the present paper he extends the investigation to the case in which the functions defining the group may be transcendental. Specifically, the group considered is the cyclic group G generated by S: $x'=kx$, $y'=\varphi(x)y$, where k is a constant with $|k|\leqq 1$, and $\varphi(x)$ is an arbitrary meromorphic function except possibly at $x=0, \infty$. By iteration one obtains for S^n with $n>0$ the transformation $x'=k^nx$, $y'=\varphi_n(x)y$, where $\varphi_n(x)=\varphi(x)\varphi(kx)\cdots\varphi(k^{n-1}x)$, with a similar expression for $n<0$. The author defines the normal domain B of the group G as follows: let $D_{1,2}$ be the set of x such that $\{\varphi_n(x)\}$, $\{\varphi_{-n}(x)\}$, $n=1, 2, 3, \cdots$, respectively, is a normal family of functions, and let D be the intersection of D_1 and D_2. B is the domain $x\epsilon D$, $y\neq 0, \infty$. It follows that G is properly discontinuous on B. Various cases are considered depending on the value of k and the behaviour of $\varphi(x)$ at $x=0, \infty$. In the final part of the paper the author constructs automorphic functions on the group G.

J. Lehner (Philadelphia, Pa.).

Bochner, Salomon. Su un teorema di Frobenius per le funzioni di Jacobi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **9**, 327–331 (1950).

Frobenius in his paper "Ueber die Grundlagen der Theorie der Jacobischen Functionen" [J. Reine Angew. Math. **97**, 16–48 (1884)] derived the fundamental inequality (*) $\sum_{\alpha,\beta=1}^{\sigma}\kappa_{\alpha\beta}\chi_\alpha\chi_\beta\geqq 0$, and he discussed conditions under which the equality sign holds. In the present paper the author studies the same question for functions on a variety S having locally complex coordinates upon which a suitable abelian Lie group of analytic homeomorphisms is defined. Under very general conditions he derives the inequality (*) and determines when the equality sign holds.

W. T. Martin (Princeton, N. J.).

Bernšteĭn, S. N. On the relation of quasi-analytic functions with weight functions. Doklady Akad. Nauk SSSR (N.S.) **77**, 773–776 (1951). (Russian)

Désignons par E_nf la meilleure approximation de $f(x)$ sur un intervalle I par des polynomes de degré n. Posons $\rho_n=\max_{p>0}(E_pf)^{1/n}p$. Soit $F(x)=\sum a_kx^k$ $(a_k>0)$ une fonction entière de degré positif. La condition (α) $f^{(n)}(x_0)=0$ $(n\geqq 0)$, $x_0\epsilon I$ et chacune des conditions (1) $E_nf\leqq[F(n)]^{-1}$, (2) $S_0=\sum\rho_n^{-1}=\infty$ suffit pour que $f(x)\equiv 0$ [S. Bernstein, Leçons sur les propriétés extrémales . . . , Gauthier-Villars, Paris, 1926]. L'auteur démontre que de (1) résulte (2). A_n étant une suite croissante, si $\sum A_k^{-1}=\infty$, il existe une fonction $f(x)\not\equiv 0$ satisfaisant à la condition (α) et telle que $\rho_n<kA_n$ (k const., $n\geqq 1$). Une fonction Φ est dite fonction de poids si, quelle que soit la fonction $H(u)$ avec $H(u)/\Phi(u)\to 0$ $(u\to\pm\infty)$, à tout $\epsilon>0$ correspond un polynome $P(u)$ tel que

$$|H(u)-P(u)|<\epsilon\Phi(u)$$

sur toute la droite. Si Φ est une fonction de poids, paire et croissante, si (β) $E_nf\leqq[\Phi(n)]^{-1}$ et si (α) est satisfait, $f(x)\equiv 0$. Si (γ) $x\Phi'(x)[\Phi(x)]^{-1}$ est une fonction croissante, une condition nécessaire et suffisante pour que de (α) et (β) résulte que $f(x)\equiv 0$ est que (δ) $S_0=\sum\lambda_n=\sum\min_{p\geqq 1}[\Phi(p)]^{1/n}p^{-1}=\infty$. Plusieurs corollaires de ce dernier théorème sont tirés, dont un résultat analogue à un théorème du rapporteur [Classes quasi-analytiques des fonctions, Moscow-Leningrad, 1937]: Si (α) et (β) ont lieu et si $\int^\infty \log\Phi(x)x^{-2}dx=\infty$, $f(x)\equiv 0$. Dans le théorème du rapporteur ce sont les coefficients de Fourier qui jouent le rôle de E_nf. *S. Mandelbrojt.*

Lelong, Pierre. Sur une propriété de quasi-analyticité des fonctions de plusieurs variables. C. R. Acad. Sci. Paris **232**, 1178–1180 (1951).

Une classe (A_n) de fonctions indéfiniment dérivables est caractérisée de la manière suivante: si $f=f(x_1, \cdots, x_n)\epsilon(A_n)$ et si f et toutes ses dérivées partielles s'annulent sur la frontière d'un domaine compact $\bar{G}$, on a $f\equiv 0$ sur $\bar{G}$. Soit $(\alpha)=(\alpha_1, \alpha_2, \cdots, \alpha_n)$, α_i entiers positifs, un indice de dérivation, et posons $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$. La fonction $f\epsilon\{M_{(\alpha)}\}$ sur $\bar{G}$ si

$$|D^{(\alpha)}f|=\left|\frac{\partial^{\alpha_1+\cdots+\alpha_n}f}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\right|\leqq AK^{|\alpha|}M_{(\alpha)}, \quad A=A_f, \quad K=K_f.$$

Posons $\mu_{|\alpha|}=\min_{(\alpha)}M_{(\alpha)}$, $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$,

$$T(r)=\max r^{|\alpha|}/\mu_{|\alpha|}, \quad I\{M_{(\alpha)}\}=\int_1^\infty \log T(r)r^{-2}dr.$$

Voici une généralisation du théorème classique de Denjoy-Carleman: Pour que la classe $\{M_{(\alpha)}\}$ soit une classe quasi analytique (A_n) il faut et il suffit que $I\{M_{(\alpha)}\}=\infty$. La quasi analyticité (A_n) peut donc être assurée en bornant une seule dérivée partielle dont l'ordre total est $|\alpha|$. Généralisation au cas où l'ensemble des dérivations est remplacé par une famille d'opérateurs $P[f]$, où P est un polynome de dérivation homogène, à coefficients constants. *S. Mandelbrojt.*

Lopatinskiĭ, Ya. B. On a generalization of the concept of analytic function. Ukrain. Mat. Žurnal 2, no. 2, 56–73 (1950). (Russian)

Let A, B be permutable linear partial differential operators ($AB=BA$) of order n, of the form

$$A=\sum a_{kl}(x,y)(\partial^{k+l}/\partial x^k\partial y^l),\quad B=\sum b_{kl}(x,y)(\partial^{k+l}/\partial x^k\partial y^l)$$

where a_{kl}, b_{kl} are analytic functions of their real arguments. A function $w=u(x,y)+iv(x,y)$ is called (A,B)-analytic if u and v are analytic functions of (x,y) and (1) $Au=Bv$, $Av=-Bu$. The function Aw is called the generalized derivative of w; it is again (A,B)-analytic. A difference quotient is defined in such a way that under certain conditions the ratio Aw_1/Aw_2 is the limit of the corresponding ratio of difference quotients. There exist uniquely determined (A,B)-analytic functions $\omega_t^{(p)}$, $p=0,1,\cdots,t=1,2,\cdots,n^2$ such that

$$A\omega_t^{(0)}=0,\quad A\omega_t^{(p+1)}=\omega_t^{(p)},$$

and at a fixed point (x_0,y_0)

$$\partial^{i_s+j_s}\omega_t^{(p)}/\partial x^{i_s}\partial y^{j_s}=\delta_{st},$$

where (i_s,j_s) is the sth pair $(1,1),(1,2),\cdots,(n,n)$. An (A,B)-analytic function regular in the neighborhood of (x_0,y_0) admits the unique expansion (analogue of a Taylor series)

$$w(x,y)=\sum_{p=0}^{\infty}\sum_{t=1}^{n^2}\omega_t^{(p)}(x,y)\cdot[A^pw]_{(x_0,y_0)}.$$

Cauchy's theorem is also generalized and so is Cauchy's formula, with the aid of a fundamental solution of (1). Example: $Au=u_{xx}-u_{yy}$, $Bu=u_{xy}$. If (1) holds, u and v satisfy $AAu=0$.

[The author states that his paper extends to $n>1$ the work of Gelbart and the reviewer [Trans. Amer. Math. Soc. 56, 67–93 (1944); these Rev. 6, 86] and Položiĭ [Doklady Akad. Nauk SSSR (N.S.) 58, 1275–1278 (1947); these Rev. 9, 507] on generalized Cauchy-Riemann equations. This is not quite correct, since in the case $n=1$ the commutativity condition $AB=BA$ excludes all interesting equations.]

L. Bers (Los Angeles, Calif.).

Theory of Series

Cândido Gomes, Marcos E. Contribution to the study of the nature of a series. Revista Ci., Lima 52, nos. 3–4, 5–8 (1950). (Portuguese)

Exposition of a well-known convergence criterion.

R. P. Boas, Jr. (Evanston, Ill.).

Vermes, P. Conservative series to series transformation matrices. Acta Sci. Math. Szeged 14, 23–38 (1951).

The author calls a matrix B a δ-matrix if it represents a series-to-series transformation with the property that convergence of $\sum u_k$ implies the existence and convergence (though not necessarily to the same limit) of the transform $B\sum u_k$. A matrix G is a β-matrix provided it represents a series-to-sequence transformation with the property that convergence of $\sum u_k$ implies existence and convergence of $G\sum u_k$. A necessary and sufficient condition for G to be a β-matrix is that $\sum_k|g_{nk}-g_{n,k+1}|<M<\infty$ and $\lim_{n\to\infty}g_{nk}$ exists for $n=0,1,2,\cdots$; and a necessary and sufficient condition for B to be a δ-matrix is that the corresponding matrix G, i.e. the matrix with (1) $g_{nk}=b_{0k}+b_{1k}+\cdots+b_{nk}$, be a β-matrix. The norm of a δ-matrix B is defined to be $\|B\|=2\{\sup_n|g_{n0}|+\sup_n\sum_k|g_{nk}-g_{n,k+1}|\}$, where B and G are related as in (1). The δ-matrices form a noncommutative complex Banach algebra with a unit element. If B is a δ-matrix and has the property $\lim_{k\to\infty}b_{nk}=0$ for each n, B is called a δ_0-matrix (every δ-matrix stronger than convergence is a δ_0-matrix). The δ_0-matrices also form a noncommutative Banach algebra with a unit element. If B and C are δ_0-matrices, and the partial sums of the series $\sum u_k$ form a bounded set, then the two series $B[C(\sum u_k)]$ and $(BC)(\sum u_k)$ are identical, and their partial sums form a bounded set.

The author exhibits some δ_0-matrices B which have the property that the relation $S(z)=f(\alpha z)$ holds in certain domains; here $S(z)$ denotes the transform by B of the Taylor series $\sum u_kz^k$ of $f(z)$, and α is a constant depending on B, but not on f or z. He also considers the δ_0-matrix $A=\sum e_kE^k$, where E^k is the matrix obtained by adjoining k columns of zeros to the identity matrix, and where $\sum e_kw^k$ represents any function $\varphi(w)$ which is holomorphic and different from zero in a region containing the closed unit disc. The matrix A evaluates no divergent series with bounded partial sums; it evaluates the series $\sum z^k$ to $\varphi(z)/(1-z)$ in the region $|z|<1$, and to zero wherever $\sum e_kz^k$ converges to zero.

G. Piranian (Ann Arbor, Mich.).

Lorentz, G. G. Direct theorems on methods of summability. II. Canadian J. Math. 3, 236–256 (1951).

The author continues his study of summability functions [Canadian J. Math. 1, 305–319 (1949); these Rev. 11, 242]. Without repeating the background of definitions given in the review cited, the content of the present paper may be described as follows (essentially as in the author's summary). All summability functions for the method $R(\lambda_n,\kappa)$, $\kappa>0$, of Riesz and the method $A(\lambda_n)$ of Abel are found under reasonable restrictions on the regularity of the sequence λ_n. Absolute summability functions are defined, necessary and sufficient conditions for such functions are found, and a description of methods which possess such functions is given. All absolute summability functions for the methods of Cesàro and Euler-Knopp are found. A final paragraph is devoted to miscellaneous applications and remarks.

J. D. Hill (East Lansing, Mich.).

Garreau, G. A. A note on the summation of sequences of 0's and 1's. Ann. of Math. (2) 54, 183–185 (1951).

Any infinite sequence of 0's and 1's determines a dyadic fraction, and so corresponds to a point of the interval (0, 1). If the set of points corresponding to the set of 0, 1 sequences, which are evaluated by a given real T-matrix A, has Lebesgue measure 1, A is said to have the Borel property. The reviewer [Ann. of Math. (2) 46, 556–562 (1945); these Rev. 7, 153] gave as a necessary condition for a real T-matrix $A=(a_{nk})$ to have the Borel property that (*) $\sum_ka^2_{nk}\to0$ as $n\to\infty$; and as a sufficient condition that (**) $\sum_n(\sum_ka^2_{nk})^q<\infty$ for some $q>0$. It was conjectured that condition (*) is not sufficient, and an example was given to show that condition (**) is not necessary. In the present paper the author gives an example to show that (*) is in fact not sufficient; and he gives a further example to show that no condition can be necessary which depends only on the rapidity with which $\sum_ka^2_{nk}$ approaches zero. [The same conclusions were reached independently by the reviewer in a paper on the Borel property to appear shortly in the Pacific Journal of Mathe-

matics.] The reviewer showed in the paper cited above that (*) implies the existence of a row-submatrix (a_{m_ik}) of (a_{nk}) which defines a method A' (not weaker than A) having the Borel property. The author shows how (**) can be used to effectively construct such a row-submatrix. He concludes with the following theorem. In order that a T-matrix $A=(a_{nk})$ have the Borel property it is necessary and sufficient that A contain a row-submatrix (a_{m_ik}) having the Borel property, and that there exist for each n an index $m_{i_n}=p_n$ such that $\sum_k |a_{nk}-a_{p_nk}|\to 0$ as $n\to\infty$. This is the first attempt to give a characterization of the Borel property. However, as a sufficient condition it would be extremely cumbersome to apply, and, as the author points out, as a necessary condition it is trivial. *J. D. Hill.*

Žak, I. E., and Timan, M. F. Absolute Abelian summability of double series. Doklady Akad. Nauk SSSR (N.S.) **78**, 849–852 (1951). (Russian)

Given a series $\sum_{0,0}^{\infty,\infty} a_{mn}$ let $\sigma_{mn}^{\alpha\beta}$ be the (C, α, β) means of it, and $g(r, \rho)=\sum_{0,0}^{\infty,\infty} a_{mn}r^m\rho^n$ $(0\leqq r<1, 0\leqq \rho<1)$ the Abel means. The series is said to be summable $|C, \alpha, \beta|$ if

$$\sum_{1,1}^{\infty,\infty} |\sigma_{mn}^{\alpha\beta}-\sigma_{m-1,n}^{\alpha\beta}-\sigma_{m,n-1}^{\alpha\beta}+\sigma_{m-1,n-1}^{\alpha\beta}|<\infty$$

and if

$$\sum_{m=1}^{\infty} |\sigma_{mn}^{\alpha\beta}-\sigma_{m-1,n}^{\alpha\beta}|<\infty,\quad \sum_{n=1}^{\infty} |\sigma_{mn}^{\alpha\beta}-\sigma_{m,n-1}^{\alpha\beta}|<\infty$$

for each n and m respectively. Similarly, the series $\sum a_{mn}$ is said to be summable $|A|$ if $\partial^2 g(r, \rho)/\partial r\partial\rho$ is absolutely integrable over the domain $0\leqq r<1$, $0\leqq\rho<1$, and if $\partial g(r, \rho)/\partial r$ and $\partial g(r, \rho)/\partial\rho$ are absolutely integrable for fixed ρ and r respectively. A series summable $|C, \alpha, \beta|$ $(\alpha>-1, \beta>-1)$ is also summable $|A|$. Summability $|A|$ implies summability A. It is shown that if at a point (x, y) the function $f(s, t)$, periodic in s and t, satisfies an analogue of the Dini-Lipschitz condition, then the Fourier series of f is summable $|A|$ at (x, y). See also Žak [Doklady Akad. Nauk SSSR (N.S.) **73**, 639–642 (1950); these Rev. **12**, 92]. *A. Zygmund* (Chicago, Ill.).

Tsuchikura, Tamotsu. On the theory of series with function terms. Tôhoku Math. J. (2) **3**, 104–113 (1951).

This paper contains generalizations to series of functions of certain theorems established by Agnew [Bull. Amer. Math. Soc. **53**, 118–120 (1947); these Rev. **8**, 456] and the reviewer [ibid. **48**, 103–108 (1942); these Rev. **3**, 147] for series of constant terms. For example, it was shown by the reviewer in the paper cited that if $\sum a_i$ is a conditionally convergent real series, then for all $x\epsilon(0, 1)$ except for a set of the first category, we have $\liminf_n \sum_1^n \alpha_i a_i=-\infty$, $\limsup_n \sum_1^n \alpha_i a_i=+\infty$, where $x=.\alpha_1\alpha_2\alpha_3\cdots$ (radix 2). The author generalizes this result as follows. Let $\sum a_i(t)$ be a series of real finite measurable functions defined in a set I of finite measure >0, and such that $\limsup_n |\sum_1^n a_i(t)|=+\infty$ almost everywhere in I. Then for all $x=.\alpha_1\alpha_2\alpha_3\cdots$ (radix 2) in $(0, 1)$ except for a set of the first category, the partial sums $\sum_1^n \alpha_i a_i(t)$ approach $+\infty$ or $-\infty$ or oscillate infinitely for almost all $t\epsilon I$. A dozen theorems of this general character are established. These are deduced from two preliminary theorems whose proofs involve adaptations of methods due to Saks [Trans. Amer. Math. Soc. **41**, 160–170 (1937)]. *J. D. Hill* (East Lansing, Mich.).

Rajagopal, C. T. A note on generalized Tauberian theorems. Proc. Amer. Math. Soc. **2**, 335–349 (1951).

Combining some known results by Minakshisundaram [Math. Z. **45**, 495–506 (1939); these Rev. **1**, 51], Pitt [Proc. London Math. Soc. (2) **44**, 243–288 (1938)] and Szász [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1930**, 315–333; Trans. Amer. Math. Soc. **39**, 117–130 (1936)], the author obtains some new Tauberian theorems for the methods of Wiener type $\Phi(t)=\int_0^\infty \phi(ut)dA(u)$, $t\to 0+$. As corollaries of more general results he proves: (i) Let $\phi(u)$ and $\psi(u)=-\phi'(u)$ be positive, decreasing and have continuous derivatives for $u\geqq 0$; further, let $\phi(0)=1$ and $\int^\infty u^{-1}\phi(u)du<+\infty$ and let $A(u)$ be of bounded variation in any finite interval with $A(0)=0$. Then

$$\liminf_{u\to\infty} u^{-1}\int_0^u A(x)dx\geqq 0$$

and the existence and boundedness of $\Phi(t)$ for $t\to 0+$ imply $\liminf_{u\to\infty} A(u)=\liminf_{t\to 0+}\Phi(t)$,

$$\limsup u^{-1}\int_0^u A(x)dx=\limsup \Phi(t).$$

(ii) If $\phi(u)$ satisfies the above regularity conditions and $\int_0^\infty u^{ix}\psi(u)du\neq 0$ for any real x, then

$$\liminf u^{-1}\int_0^u xdA(x)>-\infty$$

and the convergence of $\Phi(t)$ for $t\to 0+$ imply

$$\lim u^{-1}\int_0^u A(x)dx=\lim_{t\to 0+}\Phi(t).$$

G. G. Lorentz (Toronto, Ont.).

Ščeglov, M. P. On the generalization of Tauber's theorem. Mat. Sbornik N.S. **28(70)**, 245–282 (1951). (Russian)

The author considers various generalizations of Tauberian theorems. Let $a_0+a_1+\cdots$ be a given series and let $f(t)=\sum a_ne^{-nt}$ be its Abel means. Let $t_m>t_{m+1}\to 0$. If $t_m/t_{m+1}=O(1)$, then the conditions $f(t_m)\to S$ as $m\to\infty$ and $a_n=o(1/n)$ (even $a_n<o(1/n)$ will do) imply the convergence of $\sum a_n$ to S. However, 1) for every sequence $t_m>t_{m+1}\to 0$ such that $t_m/t_{m+1}\neq O(1)$ there is a series $\sum a_n$ with terms $o(1/n)$ such that $f(t_m)$ tends to a finite limit and yet $\sum a_n$ diverges. 2) Suppose that $a_n<O(1/n)$ and that $\sum a_ne^{-nt}$ converges for $t>0$. Suppose also that $f(t_m)\to S$ where S is a finite number and t_m is a decreasing-to-zero sequence satisfying the following conditions: a) $\liminf q_m=r$, $\limsup q_m=R$, with $q_m=t_m/t_{m+1}$, $1=r<R<\infty$; b) there is a subsequence $\{t_{m_s}\}\subset\{t_m\}$ for which

$$\liminf_{s=2,4,\cdots} t_{m_s}/t_{m_{s+1}}>1,\quad \limsup_{s=1,3,\cdots} t_{m_s}/t_{m_{s+1}}<\infty;$$

c) $\lim q_m=1$ for $m\to\infty$ and $m_s\leqq m<m_{s+1}$ $(s=2, 4, \cdots)$. Then $\sum a_n$ converges to S. 4) Let t_m be any positive sequence satisfying $t_m/t_{m+1}>r$, where $r>r_0=\frac{1}{2}(e^{-\frac{1}{2}}-e^{-1})^{-2}$. Then there is a series $\sum a_n$ with terms $O(1/n)$ such that $f(t_m)\to 1$, and yet $\sum a_n$ diverges. The problem of whether the result holds for $r_0=1$ is formulated and left open. Let d_s and D_s be the limits of indetermination of the partial sums of $\sum a_n$, and d_φ and D_φ the limits of indetermination of the Abel means $f(t)$, for $t\to 0$. The author investigates in great detail various actual possibilities in the obvious relations $d_s\leqq d_\varphi\leqq D_\varphi\leqq D_s$ under one of the following assumptions $a_n=o(1/n)$, $O(1/n)$, $O(\omega_n/n)$, where $\omega_n<\omega_{n+1}\to\infty$, $\omega_n=o(n)$. The results are too long to be given here. *A. Zygmund* (Chicago, Ill.).

Fourier Series and Generalizations, Integral Transforms

Maia, Luiz Paulo M. Introduction to the study of Fourier series. Revista Científica 1, no. 1, 1–8 (1950). (Portuguese)

Maia, Luiz Paulo M. Fourier series. II. Revista Científica 1, no. 2, 1–11 (1950). (Portuguese)

Maia, Luiz Paulo M. Fourier integrals. Revista Científica 1, no. 3, 23–24 (1950). (Portuguese)

Expository article. *R. P. Boas, Jr.* (Evanston, Ill.).

Stečkin, S. B. On the order of the best approximations of continuous functions. Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, 219–242 (1951). (Russian)

The main results of the paper have appeared previously without proof [see Doklady Akad. Nauk SSSR (N.S.) 65, 135–137 (1949); these Rev. 10, 529]. *A. Zygmund.*

Chandrasekharan, K. On the summation of multiple Fourier series. IV. Ann. of Math. (2) 54, 198–213 (1951).

Let $f(x_1, x_2, \cdots, x_k)$ be a function of period 2π with respect to each x_j, and of the class L_1. For fixed $(x_1, \cdots, x_k)$ let

$$S^\delta(R)=\sum_{\nu\leqq R}(1-\nu^2/R^2)^\delta a_{n_1\cdots n_k}e^{i(n_1x_1+\cdots+n_kx_k)},$$

where $\nu^2=n_1^2+\cdots+n_k^2$, be the spherical means of the Fourier series of f. Let

$$f(t)=f(t,x)=\frac{\Gamma(\frac{1}{2}k)}{(2\pi)^{k/2}}\int_\sigma f(x_1+t\xi_1, \cdots, x_k+t\xi_k)d\sigma_\xi$$

be the spherical averages of f at the point x. Here σ denotes the sphere $\sum\xi_i^2=1$ and $d\xi_k$ its $(k-1)$-dimensional volume element. For $p>0$ we denote the pth spherical averages of f at x by the formula

$$f_p(t)=\frac{2}{B(p,\frac{1}{2}k)t^{2p+k-2}}\int_0^t(t^2-s^2)^{p-1}s^{k-1}f(s)ds.$$

Completing previous results [Proc. London Math. Soc. (2) 50, 210–222 (1948); Ann. of Math. (2) 49, 991–1007 (1948); these Rev. 10, 113, 248], the author proves among others the following results. (1) If $f_p(t)=o((\log 1/t)^{-1})$ for some $p>0$ and for $t\to 0$, then $S^\delta(R)=o(1)$ for $R\to\infty$ and $\delta=p+\frac{1}{2}(k-1)$. (2) If $f_p(t)=o(t^\alpha)$, then $S^\delta(R)=o(1)$ for $\delta=p+\frac{1}{2}(k-1)-\theta$, with $\theta=\alpha(p-h)/(1+h+\alpha)$, h being the greatest integer $<p$. (3) If $p>0$, $\alpha\geqq 0$, then

$$f_p(t)=o(t^\alpha/|\log t|),$$

implies $S^\delta(R)=o(R^{-\alpha})$ for $\delta=p+\frac{1}{2}(k-1)+\alpha$. (4) If $p>0$, $\alpha\geqq 0$, then $f_p(t)=o(t^\alpha)$ implies $S^\delta(R)=o(R^{-\alpha})$ for

$$\delta>p+\tfrac{1}{2}(k-1)+\alpha$$

and $S^\delta(R)=o(R^{-\alpha}\log R)$ for $\delta=p+\frac{1}{2}(k-1)+\alpha$. There is a parallel set of converse results in which estimates for $S^\delta(R)$ lead to estimates for $f_p(t)$. Finally, (5) if $f_p(t)\geqq A$ near $t=0$, then $S^\delta(R)$ tends to a limit either for every $\delta>p+\frac{1}{2}(k-1)$ or for no $\delta>0$. A necessary and sufficient condition for the former is the existence of $\lim_{t\to 0} f_{p+1}(t)$. *A. Zygmund* (Chicago, Ill.).

★Maak, Wilhelm. Fastperiodische Funktionen. Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band LXI. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1950. viii+240 pp. 21.60 DM; bound, 24.60 DM.

This is a book about almost periodic functions on groups. It gives an introduction to the theory from definition [in a previously published version of the author, Abh. Math. Sem. Univ. Hamburg 16, nos. 3–4, 56–71 (1949); these Rev. 11, 327] to approximation theorem, and a certain amount of material on compact Lie groups, mainly the theorems that a compact locally Euclidean group is a Lie group and that on a semisimple Lie group all almost periodic functions are continuous. There is also an appendix on "further reading," as it were, but it is somewhat selective. The theory of "ordinary" almost periodic functions on Euclidean space is fully worked in, with sections especially devoted to them, but by and large only as far as fundamentals are concerned. Topics of details, of which there are many, are not present. A notable omission, affecting not only the Euclidean case, is the total lack of reference to, or even mention of, connections with positive-definite functions, and the reviewer thinks that for the study of subtler aspects of the theory such must not be omitted. However, it is a good informative book, and the author is justified in expecting that it will be accessible to a student without heavy prerequisites. *S. Bochner* (Princeton, N. J.).

Safronova, G. P. On a method of summation of nonsingular integrals. Doklady Akad. Nauk SSSR (N.S.) 78, 1101–1104 (1951). (Russian)

The Fourier integral $\int_0^\infty du \int_{-\infty}^{+\infty} f(t)\cos u(t-x)dt$ is said to be summable J at a point x to value s if the integral

$$J_\lambda(x)=\frac{96}{\pi\lambda^3}\int_{-\infty}^{+\infty} f(t)\frac{\sin^4\frac{1}{2}\lambda(t-x)}{(t-x)^4}dt$$

exists for all λ, and if $J_\lambda(x)\to s$ as $\lambda\to\infty$. It is shown that the method J is equivalent to $(C, 3)$. If $f(t)/(1+t^4)\epsilon L(-\infty, +\infty)$, then $J_\lambda(x)\to f(x)$ at every point x at which $f(x)$ is finite and is the derivative of the integral of f. See also Safronova, Doklady Akad. Nauk SSSR (N.S.) 73, 277–278 (1950); these Rev. 12, 94. *A. Zygmund* (Chicago, Ill.).

★Sneddon, Ian N. Fourier Transforms. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1951. xii+542 pp. $10.00.

This book deals with the application of Fourier, Mellin, Laplace and Hankel transforms to the solution of problems in physics and engineering. Chapter headings are: (I) Fourier transforms. (II) Hankel transforms. (III) Finite transforms. (IV) Theory of vibrations. (V) The conduction of heat in solids. (VI) The slowing down of neutrons in matter. (VII) Hydrodynamic problems. (VIII) Applications to atomic and nuclear physics. (IX) Two-dimensional stress systems. (X) Axially symmetric stress distributions. Appendix A: Some properties of Bessel functions. Appendix B: Approximate methods of calculating integral transforms. Appendix C: Tables of integral transforms.

The first three chapters contain the basic properties of integral transforms: inversion formulae, Faltung theorems, formulae for transforms of derivatives, etc. There are also accounts of the solution of ordinary differential equations by Laplace transforms, and of "dual" integral equations [E. C. Titchmarsh, Introduction to the theory of Fourier integrals, Oxford, 1937, §11.16]. Chapter III is devoted to finite Fourier and Hankel transforms introduced respectively by Doetsch [Math. Ann. 112, 52–68 (1935)] and the author [Philos. Mag. (7) 37, 17–25 (1946); these Rev. 8, 265].

The remaining chapters deal with applications of transform methods mainly to boundary value and initial value problems for partial differential equations. These applica-

tions are grouped into chapters according to their physical background, and each chapter includes a discussion of this background, and a derivation of the fundamental equations. Chapters IV and V treat classical problems: vibrations of strings, membranes, heavy chains, elastic beams and plates, and heat conduction. They are therefore mainly concerned with hyperbolic and parabolic partial differential equations, including $\nabla^2 u = c^{-2}u_{tt}$, $\nabla^4 u + a^{-2}u_{tt} = 0$, and $\nabla^2 u = k^{-1}u_t$. Features of the treatment are the use of a transform in the space variables in preference to a Laplace transform with respect to t, and the extensive use of finite transforms. The latter, as the author points out, is equivalent to the classical method of expanding in Fourier or Fourier-Bessel series. Chapter VI includes an account of the Wiener-Hopf method; most of the other problems in this chapter are mathematically similar to those in Chapter V. In Chapter VII, classical matters are again discussed: problems in potential flow, surface waves, and slow viscous flow, usually leading to boundary value problems for elliptic equations (notably $\nabla^2\phi = 0$ and $\nabla^4\phi = 0$). There is a section on Kelvin's principle of stationary phase. The material in Chapter XIII is somewhat heterogeneous; perhaps the most interesting topic as regards the technique of integral transforms is the theory of cosmic ray showers, which involves the solution of integro-differential equations by Mellin and Laplace transforms. The last two chapters (largely based on published work of the author) deal with boundary value problems in elasticity, the underlying equations or systems of equations being of elliptic type (e.g. $\nabla^4 X = 0$). Some of the problems in these chapters are solved by fairly straightforward use of Fourier or Hankel transforms, often involving quite formidable calculations. There are also substantial sections in which mixed boundary value problems are solved by reducing them to dual integral equations. Appendix B includes accounts of the method of steepest descents and of numerical methods for evaluating trigonometric integrals.

The book is distinguished from existing textbooks on operational methods both by its more "applied" flavour and by its much wider scope. It does not confine itself merely to the Laplace transform, and many of the applications are of a more advanced nature than is usual—the later chapters are based almost entirely on work published within the last ten years. The exposition is lucid and on the whole accurate, and answers to problems are often evaluated numerically and illustrated by diagrams. The problems are well chosen to illustrate various points of technique in using transform methods, although there might be more examples on wave propagation (transmission lines, sound waves, wave guides), and an account of the Wiener-Hopf technique applied to mixed boundary value problems [for literature cf. B. B. Baker and E. T. Copson, The Mathematical Theory of Huygens' Principle, 2nd ed., Oxford, 1950, ch. V; for a review of the 1st ed. see these Rev. 1, 315] would have been useful. A more serious omission, in the reviewer's opinion, is that the use of Laplace transforms to obtain asymptotic solutions [cf. H. S. Carslaw and J. C. Jaeger, Operational Methods in Applied Mathematics, 2nd ed., Oxford, 1948, ch. XIII] is not described. This is particularly noticeable in Chapters IV and V, where the author prefers to use finite transforms to obtain series expansions. These expansions are often slowly convergent, and it is surely useful to know that contour integral solutions can often be manipulated to put the solution into a more suitable form.

It is natural that in a book of this kind the analysis used should be mainly formal, but the reviewer considers that the treatment of divergent Fourier integrals by Abel summability should have been explained and motivated more carefully (or avoided by using a complex variable Fourier transform). Moreover, on at least two occasions this procedure leads to solutions which are meaningless because they contain properly divergent integrals: Eq. (91), p. 425 and Eq. (84), p. 124 (in the latter the sign and range of integration are also incorrect). Some theorems are not stated accurately: e.g. Th. 26 (p. 74) should refer to the open interval $(0, \pi)$, Th. 28 (p. 80) should say what is meant by "Dirichlet's conditions" for a function of two variables, and Th. 31 (p. 83) should contain the restrictions $\mu \geqq -\frac{1}{2}$, $ha + \mu > 0$.

The above remarks are intended as criticisms only of detail, not of the book as a whole. The wealth of material, much of it inaccessible in textbooks, is certain to make it useful and interesting both to specialists in many branches of science and to those who are interested in transform methods. *G. E. H. Reuther* (Manchester).

Hartman, Philip, and Wintner, Aurel. On the behavior of Fourier sine transforms near the origin. Proc. Amer. Math. Soc. 2, 398–400 (1951).

Let $f(x)$ for $x > 0$ decrease monotonically to 0 as $x \to \infty$, and be such that $\int_{+0}^{1} x f(x)dx < \infty$. Then its sine transform

$$F(t) = \int_0^\infty f(x) \sin tx \, dx$$

satisfies $\lim_{t\to 0} F(t)/t = \int_0^\infty x f(x) dx$, where both sides of the equality may be infinite. *J. L. B. Cooper* (Cardiff).

Widder, D. V. Symbolic inversions of the Fourier sine transform and of related transforms. J. Indian Math. Soc. (N.S.) 14, 119–128 (1950).

Soit $E(s) = \exp(qs)s^r + \prod_1^\infty (1-(s/a_k)) \exp(s/a_k)$ (avec q, a_k réels, $\sum_1^\infty a_k^{-2} < +\infty$, et r entier $\geqq 0$). Si $D = d/dx$, si $\exp(aD)$ est la translation $f(x) \to f(x+a)$, l'opération $E(D)$ sera définie par $E(D) = \exp(qD)D^r \lim_{n\to\infty} \prod_1^n (1-(D/a_k)) \exp(D/a_k)$. Si I est un intervalle (a, b) dans lequel $E(s)$ est sans zéros, $(1/E(s))$ est, pour $\Re(s) \epsilon I$, transformée de Laplace bilatérale d'une fonction $G(t)$: $1/E(s) = \int_{-\infty}^{+\infty} \exp(-st)G(t)dt$,

$$G(t) = (1/2i\pi) \int_{c-i\infty}^{c+i\infty} \exp(st)(1/E(s))ds, \quad c \epsilon I.$$

Soit alors $M(s)$ une fonction méromorphe $E(s)/F(s)$, F ayant une définition analogue à celle de E; soit I un intervalle où E et F sont sans zéros. Si $M(s)$ vérifie des conditions telles qu'elle soit, pour $\Re(s) \epsilon I$, une transformée de Laplace bilatérale d'une fonction $K(t)$ (soit: $\int_{-\infty}^{+\infty} |M(\sigma + i\tau)| d\tau < +\infty$, pour tout $\sigma \epsilon I$; $\lim_{|\tau|\to\infty} M(\sigma + i\tau) = 0$, uniformément pour σ dans tout compact de I), et si alors f est une transformée de φ par composition avec K: $f(x) = \int_{-\infty}^{+\infty} K(x-t)\varphi(t)dt$, intégrale absolument convergente, on a la formule d'inversion: $\varphi(x) = F(D) \cdot \int_{-\infty}^{+\infty} G(x-t)f(t)dt$. L'intégrale est absolument convergente et l'opération $F(D)$ converge en tout point x, φ étant suposée continue. Ces formules sont évidentes du point de vue "formel", il faut démontrer la convergence des opérations exécutées. (Formellement, la composition avec G est $1/E(D)$, la composition avec K est $M(D) = E(D)/F(D)$.)

Comme application, on en déduit une formule d'inversion de la formule de Fourier. Si $g(x) = (2/\pi)^{\frac{1}{2}} \int_0^\infty \sin(xt)\psi(t)dt$, alors si on forme $L(x) = -(2/\pi)^{\frac{1}{2}} \int_0^\infty (g(xt)/t(1+t^2))dt$, on a

$$\psi(t) = \lim_{k\to\infty} \left[\frac{(-x)^{k+1}}{k!} \left(\frac{L(x)}{x} \right)^{(k-1)} \right]_{x=k/t}.$$

L. Schwartz (Nancy).

Widder, D. V. Weierstrass transforms of positive functions. Proc. Nat. Acad. Sci. U. S. A. **37**, 315–317 (1951).

The author establishes for the Weierstrass transform a result analogous to the Hausdorff-Bernstein-Widder theorem for the Laplace transform. Necessary and sufficient conditions that

$$f(x)=\int_{-\infty}^{\infty}e^{-(x-y)^2/4}d\alpha(y),$$

where the integral converges for all x and $\alpha(y)$ is nondecreasing are: (1) $f(x)$ is entire; (2) $f(x+iy)=O(e^{-y^2/4})$ uniformly in $-R\leqq x\leqq R$ for every $R>0$; (3) $e^{-tD^2}f(x)\geqq 0$, $0<t<1$, $-\infty<x<\infty$. Here

$$e^{-tD^2}f(x)=t^{-\frac{1}{2}}\int_{-\infty}^{\infty}e^{-y^2/4t}\left\{\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}y^{2k}f^{(2k)}(x)\right\}dy.$$

I. I. Hirschman, Jr. (St. Louis, Mo.).

Hirschman, I. I., Jr., and Widder, D. V. On the products of functions represented as convolution transforms. Proc. Amer. Math. Soc. **2**, 97–99 (1951).

Cet article fait suite à d'autres [Trans. Amer. Math. Soc. **66**, 135–201 (1949); **67**, 69–97 (1949); ces Rev. **11**, 350]. Soit $E(s)=\exp(bs)\prod_{k=1}^{\infty}(1-(s/a_k))\exp(s/a_k)$, b réel, a_k réels $\neq 0$, $\sum_{k=1}^{\infty}a_k^{-2}<\infty$. Soit $G(t)=(1/2i\pi)\int_{-i\infty}^{i\infty}(1/E(s))\exp(st)ds$. Si G est d'un certain classe I, le second article cité ci-dessus donne la condition nécessaire et suffisante pour qu'une fonction $f(x)$ soit de la forme $f(x)=\int_{-\infty}^{+\infty}G(x-t)d\alpha(t)$, α non décroissante. Alors: si f et g sont de cette forme, il en est de même de $f(k_1x)g(k_2x)$, si k_1 et k_2 sont >0 et $k_1+k_2\leqq 1$. Si maintenant $f(x)=\int_{-\infty}^{\infty}G(x-t)\exp(-c_1t)\varphi(t)dt$, et

$$g(x)=\int_{-\infty}^{+\infty}G(x-t)\exp(-c_2t)\psi(t)dt,$$

avec $\varphi\epsilon L^p$, $\psi\epsilon L^q$, c_1 et c_2 majorant strictement tous les a_k qui sont <0, et étant majoré strictement par tous les a_k qui sont >0, alors, si k_1 et k_2 sont >0, $k_1+k_2\leqq 1$, on a $f(k_1x)g(k_2x)=\int_{-\infty}^{\infty}G(x-t)\exp(-ct)\chi(t)dt$, avec $c=k_1c_1+k_2c_2$, $\chi\epsilon L^r$ $((1/p)+(1/q)=(1/r))$, et on a la majoration

$$\|\chi\|_r\leqq E(c_1)^{-1}E(c_2)^{-1}k_1^{-1/p}k_2^{-1/q}\|\varphi\|_p\|\psi\|_q.$$

Ces formules sont appliquées à la transformation de Stieltjes. *L. Schwartz* (Nancy).

Pollard, Harry. The closure of translations in L^p. Proc. Amer. Math. Soc. **2**, 100–104 (1951).

The translates of a function $k\epsilon L^p(-\infty,\infty)$, $p>1$, span L^p if (and only if) the convolution equation

$$k*\varphi=\int_{-\infty}^{\infty}k(x-y)\varphi(y)dy\equiv 0$$

has no solution in $L^q(q=p/(p-1))$ except $\varphi\equiv 0$. If $k\epsilon L^1\cap L^p$, the author shows that the above condition will follow if $\varphi\equiv 0$ is the only function of L^q such that $\int_{-\infty}^{\infty}e^{-\sigma|x|}e^{ixt}\varphi(x)dx\to 0$ as $\sigma\to 0$ for every t which is not a zero of the Fourier transform of k. If $k\epsilon L^p$ and $|x|^{1/p}k(x)$ is summable the author proves conversely that any non-zero function $\varphi\epsilon L^q$ satisfying this latter condition also satisfies the convolution equation $k*\varphi\equiv 0$, so that under this hypothesis the two conditions are exactly equivalent. *L. H. Loomis.*

Bose, S. K. Some new properties of generalised Laplace transform. Bull. Calcutta Math. Soc. **42**, 199–206 (1950).

[For the author's earlier work on the same subject see J. Indian Math. Soc. (N.S.) **14**, 29–34 (1950); these Rev. **12**, 256. and the reference cited in the review.] In this paper the author uses recurrence formulas and other simple functional equations for Whittaker functions to derive corresponding relations for the transforms. Examples illustrate the general results. *A. Erdélyi* (Pasadena, Calif.).

Bose, S. K. On Whittaker transform. Bull. Calcutta Math. Soc. **42**, 207–212 (1950).

In addition to results of the same type as in the paper reviewed above, this paper contains two results, essentially on the combination of Whittaker transforms with Laplace or Hankel transforms. *A. Erdélyi* (Pasadena, Calif.).

***Lur'e, A. I. Operacionnoe isčislenie i ego priloženiya k zadačam mehaniki. [Operational Calculus and its Applications to Problems of Mechanics].** 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 432 pp.

This is one of a series of books on mathematics and physics for the use of engineers, and accordingly it is essentially a textbook on the applications of operational calculus to engineering problems, especially to problems of mechanical engineering. The mathematical part is slight, but there are numerous applications, treated rather fully and illustrated by examples sometimes carried through to numerical details. For all the applications encountered in this book it is sufficient to consider operational images of sectionally continuous functions which are dominated by an exponential function $M\exp(s_0t)$. For such a function, $f(t)$, the operational image is

$$F(p)=p\int_0^{\infty}e^{-pt}f(t)dt$$

where $p=s+i\sigma$ is a complex parameter and the integral is certainly absolutely convergent when $s>s_0$. The notation $F(p)\dot{\to}f(t)$ or $f(t)\dot{\leftarrow}F(p)$ is used for this relationship. A brief summary of the contents will indicate the scope of the book.

Chapter I. Definition of the operational image, and the so-called rules of operational calculus, expansion theorems, evaluation of a considerable number of operational images, Heaviside's unit function and its application to the evaluation of the operational images of various types of discontinuous and jump functions. Chapter II. Applications to (ordinary) linear differential equations with constant coefficients, systems of such equations, and to difference equations. Chapter III. Applications to problems of mechanics and elasticity which lead to a single differential or difference equation. Chapter IV. Mechanical systems of a finite degree of freedom (leading to systems of ordinary differential equations). Chapter V. Systems of infinite degree of freedom (leading to partial differential equations). Chapter VI. The complex inversion formula and its applications. Chapter VII. Miscellaneous applications, including the solution of integral and integro-differential equations by means of the operational calculus, and transversal vibrations of beams and plates. *A. Erdélyi* (Pasadena, Calif.).

***Wagner, Karl Willy. Operatorenrechnung und Laplacesche Transformation nebst Anwendungen in Physik und Technik.** 2d ed. Johann Ambrosius Barth, Leipzig, 1950. xviii+471 pp. (1 plate). DM 42.80.

This book is written primarily for engineers and physicists. The many physical applications, a majority of which are in the field of electrical engineering, are distributed throughout the book. The author frequently supplements the formal solutions of his applied problems by discussions of interesting physical aspects of the solutions. The book

begins with a chapter on Heaviside's form of operational calculus. That chapter is followed by one on the Laplace transformation which emphasizes the advantages of this form of the operational calculus and which serves as the basis for most of the rest of the book. The contents of the chapters that follow include expansion theorems, problems in ordinary and partial differential equations, special types of Volterra integral equations, electrical networks and other examples. The appendices include a table of 184 transform pairs and a bibliography. In the more theoretical parts the conditions of validity of the formulas are not always stated; for example, it is not explicitly stated that the function must be continuous if the basic formula on page 48 for the transform of the derivative is to be true. The absence of parentheses following some of the symbols for functions, as on page 455 of his tables, is somewhat confusing. This revised edition is not substantially different from the first edition [Leipzig, 1940; these Rev. 2, 134]. *R. V. Churchill.*

Polynomials, Polynomial Approximations

Sz.-Nagy, Gyula (Julius). Sur un théorème de M. Biernacki. Ann. Soc. Polon. Math. 23, 224–229 (1950).

Let A and B be arbitrary real or complex numbers; $m_1, m_2, \cdots, m_n$ arbitrary positive numbers, and

$$f(z)=\prod(z-z_k), \qquad k=1, 2, \cdots, n,$$

and $g(z)=f(z)(A+B\sum m_k/(z-z_k))$, $k=1, 2, \cdots, n$. If the circle $K\colon |z-z_0|\leqq r$ contains all the zeros of $f(z)$, then the circle $K_1\colon |z-z_0|\leqq r_1=r\sqrt{2}$ contains at least $(n-1)$ zeros of $g(z)$ as well as all the zeros of $G(z)=f(z)g'(z)-g(z)f'(z)$. This is a theorem which was essentially first proved by M. Biernacki and then by J. Dieudonné and later generalized by the reviewer [Trans. Amer. Math. Soc. 66, 407–418 (1949); these Rev. 11, 102]. The author gives a new elementary proof of the original theorem based upon the properties of the equilateral hyperbola which passes through a pair of zeros of $g(z)$. *M. Marden* (Milwaukee, Wis.).

Specht, Wilhelm. Untersuchungen über die Wurzelverteilung algebraischer Gleichungen. Math. Nachr. 4, 126–149 (1951).

The author investigates the "probability" that a complex number be a zero of a polynomial of a given class K:

$$f(x, c)=x^n-c_1x^{n-1}+c_2x^{n-2}-\cdots+(-1)^nc_n =(x-w_1)(x-w_2)\cdots(x-w_n),$$

or that a finite set of complex numbers be simultaneously the zeros of such a polynomial. Corresponding to class K is the $2n$-dimensional point set $(c_1, c_2, \cdots, c_n)$ which is also denoted by K and which is assumed to be closed and bounded and to possess a positive $2n$-dimensional content $J_{2n}(K)$. It is also assumed that every intersection of K with a $2k$-dimensional hyperplane is closed and bounded and possesses a positive $2k$-dimensional content. The region $W^*=W^*(K)$ of the complex plane containing all the zeros of all the polynomials belonging to K is likewise bounded and closed and possesses a positive 2-dimensional content. Let W be a subregion of W^* with a positive content $J_2(W)$ and let $K^k(W)$ be the subclass of K comprised of polynomials having exactly k zeros in W. For every choice of $W\subset W^*$ and every k, $0\leqq k\leqq n$, $K^k(W)$ has a positive content $J_{2n}(K^k(W))$. The quotient $p_k(W)=J_{2n}(K^k(W))/J_{2n}(K)$ is defined as the "probability" that an equation of class K has exactly k zeros in W. Thus, if $L^k(W)=K^k+K^{k+1}+\cdots+K^n$ and $M^k(W)=K^0+K^1+\cdots+K^k$, then

$$q_k(W)=J_{2n}(L^k(W))/J_{2n}(K)$$

and $r_k(W)=J_{2n}(M^k(W))/J_{2n}(K)$ are respectively the probabilities of at least k and of at most k zeros of each polynomial of K lying in W. The author proves the following: Let $\Delta(w)=\prod(w_j-w_i)^2$ with $i<j$ be the discriminant of $f(x, c)$; let H_m be the $2n$-dimensional region comprised of all points $(w_1, w_2, \cdots, w_n)$ such that the zeros $w_1, w_2, \cdots, w_m$ lie in W, and let $h_m=\int_{H_m}|\Delta(w)|^2dw$, $m=0, 1, \cdots, n$. Then

$$p_k(W)=\sum_{m=k}^{n}(-1)^{k+m}C(n, m)C(m, k)(h_m/h_0),$$

$$q_k(W)=\sum_{m=k}^{n}(-1)^{k+m}C(n, m)C(m-1, k-1)(h_m/h_0),$$

$$r_k(W)=1-\sum_{m=k}^{n}(-1)^{k+m}C(n, m+1)C(m, k)(h_{m+1}/h_0),$$

where $C(m, k)$ denotes the binomial coefficient.

M. Marden (Milwaukee, Wis.).

Brauer, Alfred. On algebraic equations with all but one root in the interior of the unit circle. Math. Nachr. 4, 250–257 (1951).

Let K be the field of rational numbers or an imaginary quadratic field. The author proves the irreducibility in K of certain types of polynomials with rational integral coefficients by proving that all but one root of the polynomials lie in the interior of the unit circle. The main results are: (1) If $\epsilon=\pm1$, the polynomial (*) $f(x)=x^n+\epsilon(a_1x^{n-1}+\cdots+a_n)$ with rational integral coefficients is irreducible in K if $a_1>a_2>\cdots>a_n>0$ except for the case

$$f(x)=x^2+a_1x+(a_1-1)=(x+1)(x+a_1-1)$$

where $a_1\geqq2$. (2) If $\epsilon=-1$, the polynomial (*) with rational integral coefficients is irreducible in K if $a_1\geqq a_2\geqq\cdots\geqq a_n>0$. From these results several other theorems are deduced, such as the following one: if $a_1>a_2>\cdots>a_{2n+1}>0$ the polynomial $x^{2n+1}\pm(a_1x^{2n}+a_2x^{2n-1}+\cdots+a_{2n+1})$ with integral rational coefficients is irreducible in K. *R. Salem.*

Touchard, Jacques. Sur une propriété des polynomes orthogonaux. C. R. Acad. Sci. Paris 232, 2279–2281 (1951).

Suppose that the c_i are the moments of a distribution $d\alpha(z)$ in the interval of integration (a, b). The forms $f_n(x, y)=\int_a^b(x-zy)^nd\alpha(z)$ form a sequence of Appell polynomials. The following property is stated here: The orthogonal polynomial $p_n(x)$ is the canonical covariant of the binary form

$$f_{2n-1}(x, y)=\sum_{i=0}^{2n-1}(-1)^i\binom{2n-1}{i}c_i\cdot x^{2n-1-i}y^i=\sum_{i=1}^{n}\lambda_i(x-\alpha_iy)^{2n-1},$$

where the λ_i are the Cotes-Christoffel numbers and the α_i are the roots of $p_n(x)$. The sequence of Appell polynomials engenders the sequence of orthogonal polynomials. Several consequences of the announced property are given.

E. Frank (Chicago, Ill.).

Fekete, Michael. On the structure of extremal polynomials. Proc. Nat. Acad. Sci. U. S. A. 37, 95–103 (1951).

This paper contains a representation for all extremal polynomials of degree n, (a) when the set S to which the

extremal polynomials belong is an arbitrary bounded, finite or infinite closed pointset in the complex plane (consisting, if finite, of $n+1$ points at least); (b) when the set S and together with it also the extremal polynomial $p(z)$ to be represented exhibit some properties of symmetry (e.g., S is symmetric in the x-axis and $p(z)$ has real coefficients only). The following is the main theorem: Let S be a bounded, finite or infinite pointset of the z-plane, consisting of at least $n+1$ points when finite and being closed when infinite. Let $C(z)=z^n+c_1z^{n-1}+\cdots+c_n$ be an extremal polynomial on S different from 0 throughout S. Then there are $m+1$ points, say $z_0, z_1, \cdots, z_m$, in S and $m+1$ positive constants, say $\lambda_0, \lambda_1, \cdots, \lambda_m$, satisfying the conditions $n\leqq m\leqq 2n$, $\sum_{\mu=0}^{m}\lambda_\mu=1$ such that $C(z)$ admits the representation: $C(z)=\Lambda(z)/M(z)$ where $\Lambda(z)$ and $M(z)$ are polynomials of the respective orders m and $m-n$ and with leading coefficients equal to 1, given by the formulas $\Lambda(z)=\sum_{\mu=0}^{m}\lambda_\mu g(z)/(z-z_\mu)$, $M(z)=\sum_{\mu=0}^{m}\lambda_\mu g(z)/C(z_\mu)(z-z_\mu)$, $g(z)=\prod_{\mu=0}^{m}(z-z_\mu)$.

E. Frank (Chicago, Ill.).

Perron, Oskar. Über die Abhängigkeit von Polynomen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1950**, 117–130 (1951).

Let f_ν be homogeneous polynomials of degree m_ν in $x_1, x_2, \cdots, x_k$. The height (Höhe) of a dependency

$$\sum C_{\lambda_0\lambda_1\cdots\lambda_k} f_0^{\lambda_0} f_1^{\lambda_1}\cdots f_k^{\lambda_k}=0 \tag{1}$$

homogeneous in the x_i is defined to be $M=m_0\lambda_0+\cdots+m_k\lambda_k$. The author proves: If the coefficients of the f_ν are indeterminates, there is a dependency (1) of height M whose coefficients are polynomials with integer coefficients in the indeterminants and where M is the product of the m_ν divided by their greatest common divisor d. Assuming the polynomials to be relatively prime and letting $m_\nu/d=p_\nu$, $M/m_\nu=l_\nu$ $(\nu=0, 1, \cdots, k)$, then $C_{l_0 0\cdots 0}$ is independent of the coefficients of f_0 and, except for a nonvanishing numerical factor, is the p_0th power of the resultant

$$R\begin{pmatrix} f_1 & f_2\cdots f_k \\ x_1 & x_2\cdots x_k \end{pmatrix}$$

and the analogues hold for other C with k zero coefficients. The author shows the above polynomial to be minimal in a sense analogous to that for a single variable.

B. W. Jones (Boulder, Colo.).

Special Functions

Parodi, Maurice. Sur quelques nouvelles conséquences d'un théorème de Laguerre. Bull. Sci. Math. (2) **75**, 41–47 (1951).

This is a sequel to an earlier paper by the same author [C. R. Acad. Sci. Paris **231**, 889–890 (1950); these Rev. **12**, 333] and contains some corollaries of the theorem referred to in the title.

A. Erdélyi (Pasadena, Calif.).

Sears, D. B. On the transformation theory of hypergeometric functions and cognate trigonometrical series. Proc. London Math. Soc. (2) **53**, 138–157 (1951).

In this paper the author continues his earlier investigations [same Proc. (2) **52**, 14–35 (1950); these Rev. **12**, 257] on the transformation theory of hypergeometric series. The principal result of the paper consists of three parts: (i) the transformation, in several possible ways, of the hypergeometric series of unit argument

$${}_{n+1}F_n(a_1, \cdots, a_{n+1}; b_1, \cdots, b_n)$$

into a linear combination of n series of the same type, (ii) a corresponding transformation of a general series which may be indicated as

$${}_nF_n(a_1, \cdots, a_n; b_1, \cdots, b_n; E)\theta_0,$$

where there is an arbitrary sequence $\theta_0, \theta_1, \theta_2, \cdots$ of real and complex numbers and E^s is the operator which converts θ_r into θ_{r+s}, and (iii) the corresponding results for the sequences $\theta_r=r!\cos r\theta$, $r!\sin r\theta$ (θ real and $\cos\theta\neq 1$). Certain exceptional cases of integer parameters are excluded, and $\sum b-\sum a$ is assumed to have a positive real part. For the validity of (ii) it is further assumed that the θ_r are such as to justify inverting the order of summation in certain double series. The proof proceeds by a very adroitly handled induction process. From the assumption that (i) holds for $n=N$, it is first proved that also (ii) must hold for $n=N$ and for suitable sequences θ_r, and from this in turn (i) is deduced for $n=N+1$. The paper also contains some results on products of hypergeometric series, on well-poised series, and, arising out of (iii), a discussion of hypergeometric series on the circle of convergence.

Reviewer's remark: The transformation (i) is closely related to one of Meijer's transformation formulae for the G-function [Nederl. Akad. Wetensch., Proc. **49**, 227–237, 344–356, 457–469, 632–641, 765–772, 936–943, 1063–1072, 1165–1175 (1946), equation (29); these Rev. **8**, 156, 379].

A. Erdélyi (Pasadena, Calif.).

Sears, D. B. On the transformation theory of basic hypergeometric functions. Proc. London Math. Soc. (2) **53**, 158–180 (1951).

The author's earlier work on transformations of hypergeometric series [same Proc. **52**, 14–35 (1950); these Rev. **12**, 257] is carried over to basic hypergeometric series. The base is p, it is assumed that $|p|<1$, and $q=p^{-1}$ is the inverse base. As in the earlier work, the author obtains rather general transformation formulae involving an arbitrary sequence $\theta_0, \theta_1, \theta_2, \cdots$. Again E is the displacement operator, $E^n\theta_r=\theta_{r+n}$. Instead of the powers of the difference operator, in the basic case the author uses two families of operators P and Q, where

$$P_0\theta_r=\theta_r, \quad P_n\theta_r=\left\{\prod_{s=1}^{n}(Ep^s-1)\right\}\theta_r$$

and $Q_n\theta_r$ is obtained when p is replaced by q. With these operators the author deduces transformations involving arbitrary sequences, and then develops rather fully the transformation theory of the basic hypergeometric series ${}_2\Phi_1$, ${}_3\Phi_2$ and of allied series.

A. Erdélyi.

Sears, D. B. Transformations of basic hypergeometric functions of special type. Proc. London Math. Soc. (2) **52**, 467–483 (1951).

The author has previously shown [see the preceding review] that many transformations of ordinary and basic hypergeometric functions may be obtained as special cases of the transformation theory of a more general class of series. These are hypergeometric series with a displacement operator substituted for the argument, applied to a sequence θ_s subject to certain convergence conditions, but otherwise arbitrary. Here the author obtains results of this type for well-poised and Saalschützian basic series. In some cases his method is identical with Bailey's [same Proc. (2) **49**, 421–435

(1947); Quart. J. Math., Oxford Ser. (1) **18**, 157–166 (1947); these Rev. **9**, 263, 92]. Typical examples, some known and others new, of basic series transformations are obtained by specializing the sequence θ_r. *N. J. Fine.*

Campbell, Robert. Contribution à l'étude des solutions de l'équation de Mathieu associée. Bull. Soc. Math. France **78**, 185–218 (1950).

Campbell, Robert. Équations intégrales des fonctions de Mathieu associées et applications. Bull. Soc. Math. France **78**, 219–233 (1950).

The associated Mathieu equation is

$$(1)\qquad \frac{d^2U}{d\xi^2}-2\nu\tan\xi\frac{dU}{d\xi}+(a+k^2f^2\sin^2\xi)U=0$$

where a, f, k, ν are constants, ν being the order. This equation arises when the wave equation is separated in spheroidal coordinates, and is a variant of the differential equation of spheroidal wave functions studied in recent years by Stratton, Morse, Bouwkamp, Blanch, and others. An associated Mathieu function (AMF) is a solution $pe_n^\nu(\xi)$ of (1) which is an entire function of ξ and reduces to Gegenbauer's polynomial $C_n^\nu(\sin\xi)$ when $k\to 0$, n being the index. For a given value of kf, such a solution exists only for certain characteristic values, a_n, of a.

In the first paper the author develops the theory of AMF's from scratch, independently of the already existing literature (much of which is not even mentioned). He uses the customary technique of expansion in series of Legendre polynomials (for order $\frac{1}{2}$), Gegenbauer polynomials (for any order and integer index), or associated Legendre functions (for any order and index), with recurrence relations for the coefficients, and an infinite continued fraction for the determination of the characteristic values of a; and it is not quite clear in which respect the author believes that his results go essentially beyond the work of previous investigators. The paper consists of three chapters devoted respectively to AMF's of order $\frac{1}{2}$, any order and integer index, any order and any (real) index.

In the second paper integral equations of the form

$$pe_n^\nu(\theta)=\lambda_n\int_{-\pi}^{\pi}N(\theta,\xi)\cos^{2\nu}(\xi)pe_n^\nu(\xi)d\xi$$

for AMF's of integer index n are investigated. Applications are given to the numerical computation of AMF's, asymptotic forms of modified AMF's for large variable, expansions in series of products of Bessel functions, and recurrence relations between AMF's of different orders. *A. Erdélyi.*

Meixner, Josef. Klassifikation, Bezeichnung und Eigenschaften der Sphäroidfunktionen. Math. Nachr. **5**, 1–18 (1951).

In this paper, an up to date compilation (without proofs) of the most important hitherto known properties of the spheroidal wave functions is given. The spheroidal wave functions are solutions of the differential equation

$$(1)\qquad (1-z^2)\frac{d^2y}{dz^2}-2z\frac{dy}{dz}+\left(\lambda-\frac{\mu^2}{1-z^2}+\gamma^2-\gamma^2z^2\right)y=0$$

which has two regular singularities at $z=\pm1$ and one essential singularity at infinity with the characteristic exponent ν. Then (2) $\cos(2\pi\nu)=f(\lambda,\mu^2,\gamma^2)$ where $f(\lambda,\mu^2,\gamma^2)$ is an entire function of the three variables λ, μ^2, γ^2. If ν and μ are fixed, λ is a function of γ and is denoted by $\lambda=\lambda_\nu^\mu(\gamma)$. Solutions of (1) can be expressed in terms of Bessel functions in the following manner:

$$(3)\qquad S_\nu^{\mu(i)}(z;\gamma)=(2\gamma z/\pi)^{-\frac{1}{2}}(z^2-1)^{-\frac{1}{2}\mu}z^\mu \times\sum_{l=-\infty}^{\infty}a^\mu_{\nu,2l}(\gamma)Z^{(i)}_{\nu+\frac{1}{2}}(\gamma z)/A_\nu^\mu(\gamma),$$

where $Z^{(i)}_{\nu+\frac{1}{2}}(\gamma z)$, for $i=1, 2, 3, 4$ respectively, denotes Bessel's, Neumann's, Hankel's first and second function of the order $\nu+\frac{1}{2}$ and argument γ^2. The $a^\mu_{\nu,2l}$ are determined by means of a three-term recurrence relation and $A_\nu^\mu(\gamma)$ is given by

$$(4)\qquad A_\nu^\mu(\gamma)=\sum_{l=-\infty}^{\infty}(-1)^l a^\mu_{\nu,2l}(\gamma).$$

Solutions of (1) involving Legendre functions are

$$(5)\qquad Ps_\nu^\mu(z;\gamma)=\sum_{l=-\infty}^{\infty}(-1)^l a^\mu_{\nu,2l}(\gamma)\mathfrak{P}^\mu_{\nu+2l}(z),$$

$$(6)\qquad Qs_\nu^\mu(z;\gamma)=\sum_{l=-\infty}^{\infty}(-1)^l a^\mu_{\nu,2l}(\gamma)\mathfrak{Q}^\mu_{\nu+2l}(z),$$

$$(7)\qquad ps_\nu^\mu(z;\gamma)=\sum_{l=-\infty}^{\infty}(-1)^l a^\mu_{\nu,2l}(\gamma)P^\mu_{\nu+2l}(z),$$

$$(8)\qquad qs_\nu^\mu(z;\gamma)=\sum_{l=-\infty}^{\infty}(-1)^l a^\mu_{\nu,2l}(\gamma)Q^\mu_{\nu+2l}(z).$$

Numerous properties of these functions, defined by (3) to (8) are listed. The paper consists of the following parts: 1) Einführung; 2) der charakteristische Exponent; 3) Definition der Sphäroidfunktionen, $S_\nu^{\mu(i)}(z;\gamma)$; 4) Eigenschaften der Sphäroidfunktionen $S_\nu^{\mu(i)}(z;\gamma)$; 5) die mit den Kugelfunktionen verwandten Sphäroidfunktionen; 6) Verknüpfungsrelationen; 7) die Separation der Wellengleichung in den Koordinaten des gestreckten und abgeplatteten Rotationsellipsoids; 8) Integralbeziehungen und Integralgleichungen; 9) weitere Entwicklungen für die Sphäroidfunktionen; 10) die γ-Asymptotik; 11) die z-Asymptotik; 12) Entwicklungen nach Sphäroidfunktionen; 13) das numerische Problem; 14) Bemerkungen zur Literatur über die Sphäroidfunktionen. *F. Oberhettinger.*

Satô, Yasuo. Transformations of wave-functions related to the transformations of coordinates systems. I. Bull. Earthquake Res. Inst. Tokyo **28**, 1–22 (1950). (English. Japanese summary)

Satô, Yasuo. Transformation of wave-functions related to the transformation of coordinates systems. II. Bull. Earthquake Res. Inst. Tokyo **28**, 175–217 (1950). (English. Japanese summary)

Typical solutions of the wave equation in Cartesian, cylindrical, and spherical polar coordinates are well known. In applied mathematics, the transition from one system of coordinates to another system is carried out frequently, and it is important to have the corresponding transformation formulae for the wave functions expressed in these coordinates. In the first of these two papers the author finds these formulae in two dimensions for a transformation from polar to Cartesian coordinates, and for a translation or rotation of polar coordinates; and in three dimensions for the transformation from spherical polar to cylindrical or Cartesian coordinates and for a translation of spherical polar coordinates. Most, if not all, of his results were already known, and the work is motivated by the author's opinion that previous proofs while "very sagacious and elegant in themselves" show a "lack of definite logical sequence of thinking". The

whole of the second paper is devoted to the problem of rotation of spherical polar coordinates. Since the radial factors are not affected by this transformation, only the transformation of surface harmonics need be considered. Here again the general form of the result is known (as pointed out by the author in a postscript), and the principal merit of the paper under review is an explicit numerical evaluation of the coefficients up to and including degree 7.

A. Erdélyi (Pasadena, Calif.).

Reĭnov, M. N. On the computation of the velocity potential of the motion of a fluid caused by the translation of an immersed body. Doklady Akad. Nauk SSSR (N.S.) **77**, 201–204 (1951). (Russian)

In Kočin's work on the wave resistance of bodies moving in water [Proceedings of the Conference on the Theory of Wave Resistance, Central. Aero-Gidrodinam. Inst., Moscow, 1937, pp. 65–134] the following function occurs in an integral equation for the velocity potential:

$$U(\alpha,\beta,\gamma)=\Re i\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\exp\left[(i\alpha\cos\theta+i\beta\sin\theta-\gamma)\sec^2\theta\right]\sec^2\theta d\theta$$
$$-\Re(2\pi)^{-1}\int_{-\pi}^{\pi}\int_0^\infty\exp\left[\lambda(i\alpha\cos\theta+i\beta\sin\theta-\gamma)\sec^2\theta\right]\times\sec^2\theta\lambda(\lambda-1)^{-1}d\lambda d\theta,$$

where the principal value is taken in the last integral. The author remarks that tables for U would facilitate computations and, to this end, expresses U in terms of two other functions for which he gives series expansions and asymptotic expansions. It is not stated whether tables are being computed.

J. V. Wehausen (Providence, R. I.).

Harmonic Functions, Potential Theory

Brelot, Marcel. Sur l'allure des fonctions harmoniques et sousharmoniques à la frontière. Math. Nachr. **4**, 298–307 (1951).

Let $\bar{R}^t$ denote the compact space obtained from Euclidean t-space, $t\geqq 3$, by the adjunction of the point at infinity, and let $u(P)$ be subharmonic in a domain $\Omega\subset\bar{R}^t$. The author uses the modern potential-theoretic tool of the "topologie fine" to obtain information about the limiting values of $u(P)$ as P approaches a regular point Q on the frontier $F\Omega$ of Ω. If we denote the limits of $u(P)$ in the topologie fine by $\lim_\varphi$, then a principal result is the following one. Let $u(P)$ be bounded and subharmonic in the open set Ω, and let $\bar{R}^t-\Omega$ be a nonpolar set. Let Q be a regular point of $F\Omega$, and let $\mathfrak{A}$ be a subset of $F\Omega$ which is "intérieurement négligeable", such that $\mathfrak{A}$ contains points of $F\Omega$, near Q, where Ω is "éffilé". Then

$$\lim_{\substack{P\to Q\\ P\in\Omega}} u(P)=\lim_{\substack{P\to Q\\ P\in F\Omega-\mathfrak{A}}}\ \lim_{\substack{M\to P\\ M\in\Omega}}{}_\varphi\ u(M),$$

which extends an earlier result due to the author [Bull. Soc. Roy. Sci. Liège **1939**, 468–477; these Rev. **1**, 122]. Then the author shows that the Perron envelopes remain unchanged if the topologie fine is used. Finally, the author considers the problem of normal derivatives at regular points of $F\Omega$, for a function harmonic in Ω; here the author proves the results announced earlier [C. R. Acad. Sci. Paris **226**, 1499–1500 (1948); **227**, 19–21 (1948); these Rev. **9**, 508; **10**, 116].

M. O. Reade (Ann Arbor, Mich.).

Lelong, Pierre. Sur les singularités complexes d'une fonction harmonique. C. R. Acad. Sci. Paris **232**, 1895–1897 (1951).

It follows from Poisson's integral that with every domain D_k in the Euclidean E_k of the variables (x_j) there is associable a domain D_{2k} containing it in the E_{2k} of the complex variables $z_j=x_j+\sqrt{-1}x_j'$ such that any function which is harmonic in D_k has a complex-analytic continuation into D_{2k}. For given D_k there must therefore exist maximal ones among the D_{2k}. The author shows that only one maximum exists and that it can be obtained in the following manner. Subtract from E_{2k} all pointsets $\sum_{j=1}^k(z_j-y_j)^2=0$ where $\{y_j\}$ varies over the boundary of D_k and in the remaining open set take the component containing D_k.

S. Bochner.

Reade, Maxwell O. On a mass distribution associated with a class of polynomials. Proc. Amer. Math. Soc. **2**, 55–63 (1951).

Soient: f une fonction sommable dans un domaine D du plan et sur tout segment rectiligne fermé de D; $L(f,x,y,r,\varphi)$ la valeur moyenne de f sur le périmètre du polygone régulier à n côtés de centre (x,y), de rayon r, d'orientation φ, contenu dans D ainsi que son intérieur; det (f,x,y,r_k,φ) le determinant d'ordre k dont la t-ième ligne est:

$$L(f,x,y,r_t,\varphi)-f(x,y),\ r_t^2,\ r_t^4,\ \cdots,\ r_t^{2k-2}.$$

Alors: pour que det $(f,x,y,r_k,\varphi)=0$ quels que soient $x,y,r_1,\cdots,r_k,\varphi$ (tels que les k polygones correspondants soient contenus dans D) il faut et il suffit que f soit un polynome k-harmonique d'un type particulier, dont le degré dépend simplement de k et n. Même énoncé en considérant des moyennes superficielles. Le résultat était connu pour $k=1$ et 2.

J. Deny (Strasbourg).

Ohtsuka, Makoto. A theorem on the Poisson integral. Proc. Japan Acad. **22**, no. 6, 195–197 (1946).

The author proves the following theorem. Let $U(z)\equiv U(re^{i\theta})$ be harmonic for $|z|<1$, and of the form

$$(1)\qquad U(re^{i\theta})=(1/2\pi)\int_0^{2\pi}U(e^{i\varphi})P(r,\theta,\varphi)d\varphi,$$

where $P(r,\theta,\varphi)$ is the Poisson kernel, and $U(e^{i\varphi})$ is Lebesgue integrable. Let G be any simply connected domain lying in $|z|<1$; let G be mapped conformally onto $|w|<1$, thus transforming $U(z)$ into $U^*(w)$, for $|w|<1$. Then $U^*(w)$ is harmonic in $|w|<1$ and can be represented in the form (1). This result should be compared with theorem 1(ii) in Evans, "The Logarithmic Potential" [Amer. Math. Soc. Colloq. Publ., vol. 6, New York, 1927, p. 46].

M. O. Reade.

Protter, M. H. On a class of harmonic polynomials. Portugaliae Math. **10**, 11–22 (1951).

On définit une nouvelle classe de polynomes harmoniques $R_{n,\nu}(x,y,z)$ $(n=0,1,2,\cdots;\nu=0,1,\cdots,2n)$ au moyen des propriétés suivantes: (1) $R_{n,\nu}$ est un polynome harmonique homogène de degré n; (2) pour toute fonction harmonique $U(x,y,z)$, régulière dans le voisinage de l'origine, on a un développement en série:

$$U(x,y,z)=\sum_{n=0}^{\infty}\sum_{\nu=0}^{2n}\frac{\alpha_{n,\nu}}{n!}R_{n,\nu}(x,y,z),$$

où les coefficientes $\alpha_{n,\nu}$ sont donnés par les formules suivantes:

$$\alpha_{n,2\nu}=\frac{\nu!(n-\nu)!}{n!}\left.\frac{\partial^n U(x,y,0)}{\partial x^\nu\partial y^{n-\nu}}\right|_{x=y=0},$$
$$\alpha_{n,2\nu+1}=\frac{\nu!(n-\nu-1)!}{n!}\left.\frac{\partial^{n-1}U_z(x,y,0)}{\partial x^\nu\partial y^{n-\nu-1}}\right|_{x=y=0}.$$

On étudie les propriétés de ces polynomes (expressions explicites, formules récurrentes, fonction génératrice, relations avec les fonctions de Legendre, de Bessel, de Lamé, de Mathieu, etc.). Des polynomes $R_{n,\nu}$ on déduit aisément d'autres polynomes $\bar{R}_{n,\nu}$ qui sont solutions de l'équation des ondes (3) $U_{xx}+U_{yy}-U_{zz}=0$. Au moyen de ces polynomes $\bar{R}_{n,\nu}$ on peut donner un procédé d'approximation de la solution du problème de Cauchy $U(x,y,0)=f(x,y)$, $U_z(x,y,0)=g(x,y)$ pour l'équation (3). *A. Ghizzetti.*

Miles, Ernest P., Jr. Certain properties of functions harmonic within a sphere. Proc. Amer. Math. Soc. 2, 213–221 (1951).

Soit u une fonction définie sur une surface sphérique S (avec le centre 0 et le rayon a), sommable sur S, et soit

$$u(\theta,\phi)\sim\sum_{n=0}^{\infty}[A_nP_n(\cos\theta)$$
$$+\sum_{m=1}^{n}(A_{n,m}\cos m\phi+B_{n,m}\sin m\phi)P_n{}^m(\cos\theta)]$$

son développement en série de fonctions sphériques. On dénotera par $U=p(u)$ la fonction harmonique (dans l'intérieur V de S) qui est donnée par l'intégrale de Poisson relative à u; par ∇U le gradient de U. En introduisant les trois nombres (finis ou $+\infty$):

$$A[U]=\int\int\int_V|\nabla U(P)|^2dV_P,$$
$$B[u]=\frac{1}{4\pi}\int\int_S dS_M\int\int_S\frac{[u(M)-u(N)]^2}{[MN]^3}dS_N,$$
$$C[u]=\sum_{n=1}^{\infty}\frac{2\pi na}{2n+1}\left[2A_n{}^2+\sum_{m=1}^{n}\frac{(n+m)!}{(n-m)!}(A^2{}_{n,m}+B^2{}_{n,m})\right],$$

on démontre les trois théorèmes suivants. (I) Si $U=p(u)$, on a $A[U]=B[u]=C[u]$. (II) Si U est une fonction harmonique dans V, pour qu'il résulte $A[U]<+\infty$, il faut et il suffit qu'il existe une fonction u, sommable sur S, telle que $B[u]<+\infty$, $U=p(u)$. (III) Soit W une fonction continue dans V avec ses dérivées de premier ordre et, pour presque tous les rayons de S, soit $W(P)\to u$ (pour $[OP]\to a$), avec u sommable sur S. On a alors $A[U]\leqq A[W]$ avec $U=p(u)$, l'égalité ayant lieu seulement si $A[U]=+\infty$ ou $W\equiv U$. *A. Ghizzetti* (Rome).

Nevanlinna, Rolf. Bemerkungen zur Lösbarkeit der ersten Randwertaufgabe der Potentialtheorie auf allgemeinen Flächen. Math. Z. 53, 106–109 (1950).

Remarks on the problem of the existence of non-constant bounded harmonic functions on a Riemann surface with a positive boundary. *M. Heins* (Providence, R. I.).

Maruhn, Karl. Einige Bemerkungen zu den Randwertaufgaben der Potentialtheorie. S.-B. Berlin. Math. Ges. 40–41, 13–28 (1942).

The paper contains four parts. I. An example shows that the second and third boundary value problems of potential theory are not necessarily uniquely solvable if the boundary data are unbounded. The author gives additional conditions which insure the uniqueness of the solution. Plemelj's solutions [Monatsh. Math. Phys. 15, 337–411 (1904)] satisfy these conditions. The proof is given in another paper [Math. Z. 48, 251–267 (1942); these Rev. 4, 277]. II. Existence theorem for the exterior third boundary value problem with unbounded data. III. The alternative for the interior third boundary value problem does not hold for a domain S, if there exists on the boundary T of S a simple layer τ whose potential vanishes identically in S. If S has this property, and S' is similar but not congruent to S, the alternative holds for S. IV. Applications to airfoil theory. *L. Bers.*

Bucerius, Hans. Die zweite Randwertaufgabe der Potentialtheorie für Kreis und Kugel. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1950, 1–11 (1951).

This paper discusses the solution of the differential equation $\Delta u+f=0$ (where f is a function defined within the circle $r=R$) with boundary condition $\partial u/\partial n=g$ on $r=R$, and the corresponding problem for a sphere. Formulas for the solutions are obtained by the use of an extended Green's function attributed to Stekloff [Ann. Sci. École Norm. Sup. (3) 19, 191–259, 455–490 (1902)] and to Hilbert [Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. 1904, 213–259]. *F. W. Perkins* (Hanover, N. H.).

Mathéev, A. Discontinuité de la derivée normale du potentiel de simple couche dans un espace de Riemann. Équation de Poisson. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. 45, 227–238 (1949). (Bulgarian. French summary)

According to the French summary the author generalizes the classical theorems on Newtonian potentials and potentials of simple and double layers to a Riemannian space, the potentials being defined by means of a fundamental solution of the Beltrami equation. *L. Bers* (Los Angeles, Calif.).

Simoda, Seturo. Sur le théorème de Müntz dans la théorie du potentiel. Osaka Math. J. 3, 65–75 (1951).

Let $G(x_1,\cdots,x_n,\xi_1,\cdots,\xi_n)=G(x,\xi)$ denote the Green function of the domain $\bar{S}$: $x_1{}^2+\cdots+x_n{}^2\leqq 1$. Let $f(x)$ satisfy in $\bar{S}$ a Hölder condition with exponent α $(0<\alpha<1)$ and constant K. Set $u(x)=\int_S G(x,\xi)f(\xi)d\xi$ $(d\xi=d\xi_1\cdots d\xi_n)$. In S the second derivatives of u are bounded in modulus by a constant depending only on u, α, K and a bound for $|f|$, and satisfy a Hölder condition with exponent α and constant MK, M depending only on u and α [C. Müntz, J. Reine Angew. Math. 139, 52–79 (1911)]. The author gives a new proof of this theorem. *L. Bers* (Los Angeles, Calif.).

Nicolescu, Miron. Le problème biharmonique pour un demi-plan. An. Acad. Repub. Pop. Române. Secţ. Şti. Mat. Fiz. Chim. Ser. A. 2, 425–442 (1950). (Romanian. French summary)

Let $g(x)$ be continuous and bounded for $-\infty<x<\infty$, let $f(x)$ be of class C'', with $f(x)$ and $f'(x)$ bounded, for $-\infty<x<\infty$. The author solves the problem of finding all real $u(x,y)$, biharmonic for $y>0$, such that $u(x,0+)\equiv f(x)$, $u_y(x,0+)\equiv g(x)$, by showing that the general solution is the sum of a certain particular solution and the general solution to an associated, more elementary, problem. The question of uniqueness is discussed and solved. *M. O. Reade.*

Differential Equations

Urabe, Minoru. Reduced forms of ordinary differential equations in the vicinity of the singularity of the second kind. J. Sci. Hiroshima Univ. Ser. A. 14, 26–37 (1949).

In the terminology of Forsyth [Theory of Differential Equations, Cambridge Univ. Press, 1900, Part II, p. 49]

the differential equation $dy/dx=f(x, y)$ in the vicinity of a singularity of the second kind $(x, y)=(0, 0)$ is considered. Using the Newton polygon construction it is shown that near $(0, 0)$ the equation $dy/dx=f(x, y)$ can be reduced, in a certain sense, to one of the following four forms:

(1) $t^k(dv/dt)=av+pt+\cdots$, $k\geqq 2$, $a\neq 0$;

(2) $t(dv/dt)=v^\lambda(av^n+pt^m+\cdots)$, $a,p\neq 0$, $n\geqq 0$, $m>0$, and $\lambda\geqq 1$ when $n=0$;

(3) $dv/dt=t^\nu v^\lambda(av^n+pt^m+\cdots)$, $a,p\neq 0$, $n\geqq 0$, $m>0$, $\nu\geqq 0$, $\lambda\geqq 1$;

(4) $t(dv/dt)=(av^n+pt+\cdots)/(bv^m+qt+\cdots)$, $a, b\neq 0$, $m\geqq 1$, $n\geqq 1$.

The dots represent terms of higher order in v and t than those displayed. *E. A. Coddington* (Cambridge, Mass.).

Erugin, N. P. On the theory of differential equations (ordinary and partial). Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 355–366 (1951). (Russian)

The author [same journal **14**, 315 (1950); these Rev. **12**, 99] has previously discussed systems of the form $dx/dt=P(x, y)$, $dy/dt=Q(x, y)$, where P and Q satisfy the Cauchy-Riemann equations. These equations may be solved by setting $z=x+iy$. He now considers equations which may be solved by means of a suitable change of variable which converts these into systems of the above type. Many examples are given. *R. Bellman.*

Wintner, Aurel. On the small divisors in integrations by Laplace transforms. Amer. J. Math. **73**, 173–180 (1951).

Soit le système d'équations linéaires $dx/dt=A(t)x$, où x est un vecteur $(x_1, \cdots, x_m)$. Supposons que la matrice des coefficients $A(t)$ soit représentée, pour $0\leqq t<\infty$, par $A(t)=\int_0^\infty e^{-ts}d\alpha(s)$, avec $[\alpha]<\infty$, où $[\alpha]$ est la plus grande variation totale des m^2 fonctions scalaires qui forment la matrice $\alpha(t)$. Supposons encore que $\int_0^\infty|d\alpha(s)|/s<\infty$. Dans ces conditions le système admet m vecteurs-solutions linéairement indépendantes: $x=x^1(t), \cdots, x=x^m(t)$, lesquels, rangés en matrice $X=(x^1, \cdots, x^m)$, peuvent être mis sous la forme $X(t)=\int_0^\infty e^{-st}d\beta(s)$, avec $[\beta]<\infty$, $\beta(+0)-\beta(0)=E$, où E est la matrice unitaire. Ainsi $X(t)=E+X_0(t)$ avec $\lim_{t\to\infty}X_0(t)=0$ $(X_0(t)=\int_{+0}^\infty e^{-st}d\beta(s))$. *S. Mandelbrojt.*

Hartman, Philip, and Wintner, Aurel. On the non-increasing solutions of $y''=f(x, y, y')$. Amer. J. Math. **73**, 390–404 (1951).

Gli autori indicano criteri perchè l'equazione

(1) $$y''=f(x, y, y')$$

ammetta soluzioni definite su tutto il semiasse $x\geqq 0$ e soddisfacenti ivi alle condizioni (2) $y(0)=y_0$, $y(x)\geqq 0$, $y'(x)\leqq 0$. P. es., le (2) possono esser soddisfatte, per ogni y_0 positivo assegnato, se $f(x, y, z)$, continua per $0\leqq x<+\infty$, $0\leqq y<+\infty$, $-\infty<z\leqq 0$, soddisfa alle $f(x, 0, 0)=0$, $f(x, y, 0)\geqq 0$ e se per ogni $c=$const.>0 esiste una funzione positiva $\varphi_c(z)$ $(-\infty<z\leqq 0)$ tale da aversi $|f(x, y, z)|\leqq\varphi_c(z)$ per $0\leqq x\leqq c$, $0\leqq y\leqq c$, $-\infty<z\leqq 0$, $\int_{-\infty}^{z}[z/\varphi_c(z)]dz=-\infty$. Se $f(x, y, z)\geqq 0$ e se le soluzioni della (1) sono univocamente determinate dai valori iniziali, le condizioni relative a $\varphi_c(z)$ possono essere soppresse, ma in generale si dovrà supporre y_0 variabile in un conveniente intorno dell'origine. Gli autori indicano anche un criterio di unicità per il problema al contorno $y''=f$, $y(a)=\alpha$, $y(b)=\beta$ e teoremi perchè le soluzioni della (1) soddisfacenti alle (2) siano infinitesime all'infinito. *G. Scorza-Dragoni* (Padova).

Wintner, Aurel. On the non-existence of conjugate points. Amer. J. Math. **73**, 368–380 (1951).

Let p, f be real continuous functions, p positive, on a t-interval I which need not be closed or bounded, and consider on I only real, not identically zero, solutions x of (1): $(px')'+fx=0$. If no solution of (1) has more than one zero on I, then (1) is called disconjugate on I. The author proves, as a consequence of Sturm's separation and comparison theorems, that (1) is disconjugate on I if and only if there exists on I some function y possessing a continuous first derivative and satisfying (2): $R(y)\leqq 0$ at every point of I, where R is the Riccati operator for (1), $R(y)=y'+y^2/p+f$. Most of the remainder of the paper is devoted to the application of this result for the case $p\equiv 1$. Certain known sufficient criteria for the disconjugate nature of (1) are shown to follow from this result, and several new sufficient conditions are given. An example is: if $(\int_t^\infty f(s)ds)^2\leqq\frac{1}{4}f(t)$, $0<t<\infty$, then (1) is disconjugate on $0<t<\infty$. If $p\equiv 1$ and $I: t_0<t<\infty$, then (1) is called oscillatory if (1) has a solution whose zeros cluster at infinity; otherwise (1) is called nonoscillatory. Sufficient conditions for (1) to be oscillatory or nonoscillatory are given. Example: if $\int^\infty f(t)dt$ is convergent and $\int^\infty \exp\{-2\int^t[\int_u^\infty f(v)dv]du\}dt<\infty$, then (1) is oscillatory. A sufficient condition for the nonexistence of an L^2 (at ∞) solution of (1) is obtained. Many of the proofs result by properly choosing a y in (2). *E. A. Coddington.*

Gol'din, A. M. On a criterion of Lyapunov. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 379–384 (1951). (Russian)

The author uses simple geometric methods to extend the well-known criterion of Liapounoff concerning the boundedness of solutions of $u''+p(t)u=0$ for periodic $p(t)$, in the case where $p(t)\geqq 0$. *M. Bellman.*

Vasil'eva, A. B. On the differentiability of the solutions of differential equations containing a small parameter. Mat. Sbornik N.S. **28(70)**, 131–146 (1951). (Russian)

Detailed proofs are given of results announced earlier [Doklady Akad. Nauk SSSR (N.S.) **61**, 597–599 (1948); these Rev. **10**, 298]. [The author does not agree with the comments of the reviewer in the cited review.] *N. Levinson* (Cambridge, Mass.).

Reeb, G. Über dynamische Systeme mit lauter periodischen Bewegungen. Abh. Math. Sem. Univ. Hamburg **17**, 98–103 (1951).

The author considers generalizations of classical approaches to the problem of periodic solutions of $x''+x=\mu f(x, x')$ for small μ to more general dynamical systems.

(a) Poincaré's method of small parameters: Let E_μ be a vectorfield, depending on μ, in a manifold V_n, such that the trajectories of E_0 fiber V_n into circles, with base space V_{n-1}. One derives from E_μ a vectorfield G in V_{n-1} (the infinitesimal deviation of E_μ from E_0, averaged along a trajectory). A singular point of G with nonzero index gives rise to periodic solutions of E_μ for all small μ. (In the classical case the trajectories of E_0 are the circles around the origin in the (x, x')-plane, and V_{n-1} can be pictured as any ray from the origin.)

(b) Relaxation oscillations: E_μ is defined in R^{2p} by $\dot{x}_i=u_i$, $\dot{u}_i=-x_i+\mu f_i(x, u)$. The trajectories of E_0 are great circles on the spheres $\sum x_i^2+u_i^2=$const. For each point P one considers the vector from P to the point where the trajectory of E_μ is after going "approximately once around". The relaxation assumption is that there is a spherical shell $m\leqq\sum x_i^2+u_i^2\leqq M$, such that the vectorfield V just de-

scribed points into the shell at every point of the boundary. It can then be shown, with the help of the Eckmann-Whitehead theorem on vectorfields that for odd p there must be a singularity of V, i.e. a closed trajectory of E_p.

In the last part the author assumes the existence of an integral invariant, and states consequences for the field G (see (a)) and the base space V_{n-1}; e.g. the even-dimensional Betti numbers are not zero. *H. Samelson.*

Ževakin, S. A. On finding the limit cycles of systems near to certain nonlinear ones. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 237–244 (1951). (Russian)

Consider the analytical system $dx_i/dt = X_i(x_1, x_2, x_3, \mu)$, $i=1, 2, 3$. It is assumed that for $\mu=0$ there is a limit-cycle Γ (not containing singular points). The author obtains rather complicated sufficient conditions in order that for μ small there exist a unique limit-cycle near Γ and tending to Γ as $\mu\to 0$. The same method is applicable to a similar system with any number n of equations. *S. Lefschetz.*

Rapoport, I. M. On the stability of oscillations of material systems. Doklady Akad. Nauk SSSR (N.S.) **77**, 25–28 (1951). (Russian)

The method given in two previous papers [same Doklady (N.S.) **73**, 889–890 (1950); **76**, 793–795 (1951); these Rev. **12**, 183, 827] is used to discuss the question of stability of the oscillations determined by the system of equations $d^2z/dt^2+\lambda^2P(t)\cdot z=0$, $P(t+T)=P(t)$, where $P(t)$ is a periodic symmetric matrix with distinct positive values in $0\leqq t\leqq T$. With suitable conditions on the elements of $P(t)$, an asymptotic formula is given for the bounds of the region of instability of the solution of the above system.

C. G. Maple (Washington, D. C.).

Malkin, I. G. On the solution of a stability problem in the case of two purely imaginary roots. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 255–257 (1951). (Russian)

Consider the analytical system of $n+2$ equations

$$(1)\qquad \begin{cases} \dfrac{dx}{dt}=-\lambda y+X, & \dfrac{dy}{dt}=\lambda x+Y, \\[2mm] \dfrac{dx_s}{dt}=p_sx+q_sy+\sum p_{si}x_i+X_i=f_s(x, y, x_1, \cdots), \end{cases}$$

where λ, p_s, q_s, p_{si} are real constants, and X, Y, X_i are real analytic functions of x, y and the x_s in a neighborhood of the origin with expansions beginning with terms of degree at least two. It is assumed that the characteristic roots, other than $\pm\lambda i$, all have negative real parts. According to Lyapunov the stability problem may be dealt with as follows. Take the auxiliary system

$$(2)\qquad (\lambda x+Y)\frac{\partial v_s}{\partial y}+(-\lambda y+X)\frac{\partial v_s}{\partial x}=f_s.$$

This system has a formal solution

$$(3)\qquad v_s(x, y)=v_s^1(x, y)+v_s^2(x, y)+\cdots,$$

where v^k is a form of degree k. Then the stability of (1) is equivalent to that of

$$\frac{dx}{dt}=-\lambda y+X(x, y, v_1, \cdots, v_n),\quad \frac{dy}{dt}=\lambda x+Y(x, y, v_1, \cdots, v_n).$$

The calculation of the forms v is rather arduous. The author shows however that once the v_s^1 are known then the calculation of the v_s^k, $k>1$, is comparatively simple.

S. Lefschetz (Princeton, N. J.).

Sim, A. C. A generalization of reversion formulae with their application to non-linear differential equations. Philos. Mag. (7) **42**, 228–238 (1951).

Author's summary: "The formulae for algebraic reversion are extended to revert a class of non-integral power series, and also generalized to revert series in which the coefficients contain operators. These formulae are shown to have a simple and powerful application to non-linear differential equations, which they expand into an infinite sequence of linear equations." From the author's conclusion: [The method] "is not usually suitable when non-linear terms are large." *N. Levinson* (Cambridge, Mass.).

Cartwright, Mary L. Forced oscillations in nonlinear systems. J. Research Nat. Bur. Standards **45**, 514–518 (1950).

In an earlier paper [Cartwright and Littlewood, J. London Math. Soc. **20**, 180–189 (1945); these Rev. **8**, 68] it was shown that for large k and certain values of b in the interval $0<b<\frac{2}{3}$ the equation $\ddot{x}=k(1-x^2)\dot{x}-x-bk\lambda\cos(\lambda t+\alpha)$ has two stable solutions of distinct periods $(2n+1)2\pi/\lambda$ and $(2n-1)2\pi/\lambda$ where n is an integer. It is shown here how an approximate form of these solutions may be conveniently obtained from very general results and the general results are proved in detail. *N. Levinson.*

Minorsky, Nicolas. Sur l'oscillateur non linéaire de Mathieu. C. R. Acad. Sci. Paris **232**, 2179–2180 (1951).

The author discusses the existence and stability of approximately periodic solutions of the differential equation $\ddot{x}+[1+\epsilon(A-Cx^2)\cos 2t]x=0$ in which A and C are constants, and ϵ is a small parameter. It is shown that no such solutions can exist if $A/C<0$. If $A/C>0$ there exist, in first approximation with respect to ϵ, stable periodic solutions. The "amplitude" $(x^2+\dot{x}^2)^{\frac{1}{2}}$ and the "phase" $\arctan(\dot{x}/x)$ of these solutions are approximately equal to A/C and to $k\pi/2$, respectively, where $k=0, 1, 2$, or 3, depending on the initial conditions. The method used is the same as in two previous notes by the author [same C. R. **231**, 1417–1419 (1950); **232**, 1060–1062 (1951); these Rev. **12**, 413, 611].

W. Wasow (Los Angeles, Calif.).

Cecconi, Jaurès. Su di una equazione differenziale non lineare di secondo ordine. Ann. Scuola Norm. Super. Pisa (3) **4**, 245–278 (1950).

The author considers the differential equations

$$(\alpha)\qquad y''+y'|y'|+qy'+y-p^2y^3=0,$$
$$(\beta)\qquad y''+y'|y'|+qy'+y-p^2y^3=r\sin\omega t,$$

where p, r and ω are positive, $q\geqq 0$, and $y'=dy/dt$. For (α) considered in the Poincaré-Bendixon (y, y')-plane the origin is stable in the strong sense that solutions near the origin tend to it as $t\to\infty$. The only other singular points, which are $(\pm 1/p, 0)$, are saddle points. The author shows that for $q=0$, $p=1$ the solutions of (α) for certain initial values, not close to $(0, 0)$, do not exist over the semi-infinite range $(t_0, +\infty)$. For the equation (β) it is assumed that $q>0$, and it is shown that for r small enough there exists a periodic solution of period $2\pi/\omega$ to which nearby solutions tend as $t\to+\infty$. [This last result, apart from the particular appraisals associated with (β), is a special case of a recently published general theorem, proved much more simply than the special case [A. B. Farnell, C. E. Langenhop, and N. Levinson, J. Math. Physics **29**, 300–302 (1951); these Rev. **12**, 706]. *N. Levinson* (Cambridge, Mass.).

Yosida, Kôsaku. On Titchmarsh-Kodaira's formula concerning Weyl-Stone's eigenfunction expansion. Nagoya Math. J. 1, 49–58 (1950).

The expansion formula for the singular self-adjoint second order linear differential operator treated by Weyl using singular integral equation theory, by Stone using Hilbert space methods and by Titchmarsh using function theoretic methods is here given a very much simpler and shorter treatment. The formula of Titchmarsh and Kodaira for the density matrix is also obtained. The results for the singular case are derived as a limiting case, in a very natural way, from the classical Sturm-Liouville expansion theorem for the nonsingular case. The Weyl notions of limit-circle and limit point play a fundamental role as does the Helly selection theorem. [An independent and very similar approach also going by a limiting process from the nonsingular to the singular case was given by the reviewer, Duke Math. J. 18, 57–71 (1951); these Rev. 12, 828.] *N. Levinson.*

Borg, Göran. Über die Ableitung der S-Funktion. Math. Ann. 122, 326–331 (1950).

Consider the differential equation (E): $y''+(\lambda-q)y=0$, where $q=q(x)$ is continuous for $0\leqq x<\infty$. If $\int_0^\infty|q(x)|dx<\infty$, then a solution $y=y(x,\lambda)$ of (E) such that $y(0,\lambda)=0$, $y'(0,\lambda)=1$ has the form

$$y(x,\lambda)=M_1(k)e^{ikx}+M_2(k)e^{-ikx}+o(1),\ x\to\infty,$$

for all $k>0$ ($k^2=\lambda$). The function $S=M_2/M_1$ is called the S-function for (E). If, in addition, $\int_0^\infty|xq(x)|dx<\infty$, then $S'=dS/dk$ exists and is continuous for $k>0$. The author investigates S' under the hypothesis $|x^2q(x)|\leqq\text{const.}, x\to\infty$. He assumes further that $q=\phi/g$ where g, g' are continuous, positive for $0\leqq x<\infty$, $x^{-2}g(x)\to1$, $(2x)^{-1}g'(x)\to1$ as $x\to\infty$, and ϕ is almost periodic, $\phi(x)=-\phi(-x)$, $\phi(x)\sim\sum A_n\sin 2\alpha_n x$, $\sum\alpha_n^{-2}<\infty$. He proves (a) S' exists and is continuous for $k\neq\alpha_n$, (b) $\lim(\log|k-\alpha_n|)^{-1}S'(k)=i(2\alpha_n)^{-1}(1+S^2(\alpha_n))A_n$, as $k\to\alpha_n$, (c) if the Fourier series of ϕ has a finite number of terms, then the set of all such functions satisfying $1+S^2(\alpha_n)=0$ for some α_n is a zero set in the set of all functions which have finite Fourier sums. The proofs of (a) and (b) depend on certain appraisals and the properties of almost periodic functions. The proof of (c) is equivalent to the fact (which the author proves) that the zeros of an analytic function (not identically zero) of n complex variables form a set of Lebesque ($2n$-real-dimensional) measure zero. An example is given to show that (a) may not hold if the α_n have a finite cluster point. Similar statements hold if $\phi(x)=-\phi(-x)$ is not assumed. *E. A. Coddington.*

Heinz, Erhard. Zur Frage der Differenzierbarkeit der S-Funktion. Math. Ann. 122, 332–333 (1950).

Under the same assumptions as in the paper reviewed above, the author proves very simply that

$$S'(\alpha_n+0)-S'(\alpha_n-0)=\pi A_n/\alpha_n$$

if $1+S^2(\alpha_n)=0$. This, together with Borg's results, prove S is nondifferentiable at every $k=\alpha_n$, and that the potential ϕ is uniquely determined by S. *E. A. Coddington.*

Spragens, W. H. On series of Walsh eigenfunctions. Proc. Amer. Math. Soc. 2, 202–204 (1951).

The author compares expansions over (0, 1) firstly in terms of eigen-functions of $u''+(\rho^2-g(x))u=0$, with $g(x)$ continuous and boundary conditions $u(0)=u(1)=0$, and secondly in terms of a Fourier sine-series. He establishes uniform equiconvergence if the expanded function is Lebesgue-integrable over $0\leqq x\leqq1$. Similar results for functions of L^2 or for different boundary conditions were given by J. L. Walsh [Ann. of Math. (2) 24, 109–120 (1923)] and A. Haar [Math. Ann. 69, 331–371 (1910)]. An extension is suggested for double series, as considered by J. Mitchell [Amer. J. Math. 65, 616–636 (1943); these Rev. 5, 96]. [Reviewer's comments. A hiatus in the proof occurs where the author states that the uniform boundedness of the difference-kernels is "readily seen". Haar [loc. cit., pp. 341–342] needed for this purpose the second term in the asymptotic expansion of the eigen-functions. The question seems to arise of whether it is sufficient for these formulae that $g(x)$ should be merely continuous.] *F. V. Atkinson* (Ibadan).

Dyubuk, A. F., and Monin, A. S. On mutually orthogonal systems of functions. Doklady Akad. Nauk SSSR (N.S.) 76, 337–340 (1951). (Russian)

The system of differential equations

$$(k_i(x)u_i'(x))'-q_i(x)u_i(x)+\lambda\rho_i(x)u_i(x)=0,\quad i=1,2,\cdots,N,$$

together with the boundary conditions $u_i'(x_i)+h_iu(x_i)=0$, $u_i'(0)-\sum_j a_{ij}u_j(0)=0$ is considered. Here $\rho_i(x)\geqq0$, $k_i(x)\geqq0$, $0<k_i(0)<\infty$, $k_i(x_i)<\infty$. For $i\neq1$, $a_{1i}\neq0$, $a_{i1}\neq0$ and $a_{1i}a_{i1}>0$. For $i<j<k$, $a_{ij}a_{jk}a_{ki}=a_{ji}a_{kj}a_{ik}$. The characteristic value problem is considered. Variational methods are used (with a reference to Courant and Hilbert, Methoden der mathematischen Physik, Bd. 1, Kap. 6 [2d ed., Springer, Berlin, 1931]) in establishing results. *N. Levinson.*

Miller, Kenneth S. A Sturm-Liouville problem associated with iterative methods. Ann. of Math. (2) 53, 520–530 (1951).

The linear differential system $Lu=u^{(n)}+\cdots=0$, $U_\alpha'=C_\alpha$ is studied on a finite interval. It is assumed that the coefficients of L are of class C^n and that $Lu=U_\alpha'=0$ is incompatible. The solution is defined as the limit of a sequence u_m and an error function as $\epsilon_m=Lu_m$. It is shown that $\epsilon_m=(I+T)\epsilon_{m-1}$, where I is the identity and T is self-adjoint and of finite norm. The function ϵ_0 can be expanded in a complete orthonormal set with characteristic numbers on $0<\lambda<1$. The normalized characteristic functions and the corresponding characteristic numbers are given explicitly. *J. M. Thomas* (Durham, N. C.).

Pini, Bruno. Sulle proprietà di minimo, e relative conseguenze, delle autosoluzioni di un sistema autoaggiunto di equazioni differenziali lineari omogenee del secondo ordine. Rivista Mat. Univ. Parma 1, 319–345 (1950).

This paper is a continuation of an earlier one of the author [Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 8, 351–377 (1949); these Rev. 11, 721], and deals with a self-adjoint two-point boundary problem involving a vector differential equation of the second order in $y(x)\equiv(y_i(x))$, $i=1,\cdots,n$, and the null end-conditions $y(a)=0=y(b)$. Extremizing properties of proper values and proper solutions are established; for systems which admit a particular type of canonical form there are obtained results on the asymptotic character of proper values and solutions. As mentioned in the review of the earlier paper, the author was evidently unaware of the extensive literature of the past two decades on boundary problems associated with the calculus of variations, and which include the considered problem as a very special case. *W. T. Reid.*

Volpato, Mario. Criteri di confronto e di unicità per le soluzioni dell'equazione $p=f(x, y, z, q)$ coi dati di Cauchy. Rend. Sem. Mat. Univ. Padova 20, 232–243 (1951).

In the real field sufficient conditions are established for the uniqueness of the solution of $p=f(x, y, z, q)$ and it is stated that the results can be extended to

$$p_1=f(x_1, \cdots, x_n; z; p_2, \cdots, p_n).$$

The conditions are rather complicated and occupy about a page and a half for the case $n=2$. They consist principally of inequalities on the variation of f under the following conditions: (i) the argument z alone is varied; (ii) the arguments z, q are varied, whereas y is replaced by a lower bound $\sigma(x)$ or by an upper bound $\tau(x)$. *J. M. Thomas.*

Germay, R. H. J. Application de la méthode des fonctions majorantes à la théorie des équations récurro-différentielles. Bull. Soc. Roy. Sci. Liège 19, 359–368 (1950).

Germay, R. H. Sur certains systèmes récurrents d'équations aux dérivées partielles du premier ordre. Bull. Soc. Roy. Sci. Liège 19, 438–447 (1950).

Germay, R. H. J. Sur certains systèmes d'équations récurrentes aux dérivées partielles du premier ordre. II. Bull. Soc. Roy. Sci. Liège 19, 503–506 (1950).

Germay, R. H. J. Application de la méthode des fonctions majorantes à la théorie des systèmes d'équations récurro-différentielles. Bull. Soc. Roy. Sci. Liège 20, 2–13 (1951).

Germay, R. H. J. Sur les équations récurrentes aux dérivées partielles du premier ordre, de forme linéaire et homogène. Ann. Soc. Sci. Bruxelles. Sér. I. 65, 25–32 (1951).

These five papers prove particular cases of the following existence theorem: In the set of holomorphic functions, the system

$$\partial u_{l\lambda}/\partial x_1=F_{l\lambda}[x_l, u_{j\mu(\lambda)}, \partial u_{k\nu(\lambda)}/\partial x_m],$$
$$u_{l\lambda}(0, x_2, \cdots, x_n)=\varphi_{l\lambda}(x_2, \cdots, x_n),$$
$$i, j, k=1, 2, \cdots, p;\ l=1, 2, \cdots, n;\ m=2, \cdots, n;$$
$$\mu(\lambda), \nu(\lambda)=\lambda, \cdots, \lambda+r;\ \lambda=1, 2, \cdots, \infty,$$

where the F's, φ's, n, p, r are given, has a unique solution. The particularizations giving the cases treated are, respectively, as follows: (i) $n=p=r=1$, the φ's are all equal; (ii) the F's do not contain the x's and are linear and homogeneous in $\partial u/\partial x$; (iii) the F's are linear in $\partial u/\partial x$; (iv) $n=1$; (v) $p=r=1$, the F's are linear and homogeneous in $\partial u/\partial x$ and do not contain the u's. It will be noted that (iii), published earlier, implies (v). It is probable that the general theorem can be proved more easily than any of the above particular cases. *J. M. Thomas* (Durham, N. C.).

Aržanyh, I. S. Integration of canonical systems of order greater than zero. Uspehi Matem. Nauk (N.S.) 5, no. 4(38), 144–153 (1950). (Russian)

The author considers what he terms "canonical systems of order r", that is, systems of differential equations: (1) $\dot{q}_i=(\partial H/\partial p_i)+(F_\alpha \partial H_\alpha/\partial p_i)$, $\dot{p}_i=-(\partial H/\partial q_i)-(F_\alpha \partial H_\alpha/\partial q_i)$, $i=1, \cdots, n$, and the summation convention is used for α, which runs from 1 to r. The author demonstrates various methods for obtaining integrals of such a system by finding integrals of appropriate systems of partial differential equations. The following generalization of the Hamilton-Jacobi procedure is typical of the results: Let the functions $H, H_1, \cdots, H_r$ be independent of t and be in involution. Let the partial differential equations:

$$H(q_1, \cdots, q_n, \partial W/\partial q_1, \cdots, \partial W/\partial q_n)=h,$$
$$H_j(q_1, \cdots, q_n, \partial W/\partial q_1, \cdots, \partial W/\partial q_n)=h_j \quad (j=1, \cdots, r)$$

be consistent and let

$$W(q_1, \cdots, q_n, h, h_1, \cdots, h_r, c_1, \cdots, c_{n-r-1})$$

be a complete integral, the c's being arbitrary constants. Then (1) has the integrals: $p_i=\partial W/\partial q_i$, $t+c=\partial W/\partial h$, $q_k=\partial W/\partial c_k$, where $i=1, \cdots, n$, $k=1, \cdots, n-r-1$. *W. Kaplan* (Ann Arbor, Mich.).

John, Fritz. The fundamental solution of linear elliptic differential equations with analytic coefficients. Comm. Pure Appl. Math. 3, 273–304 (1950).

Fundamental solutions of linear homogeneous elliptic partial differential equations of order m for an unknown function of n variables have been studied in various special cases by many mathematicians (E. E. Levi, Hadamard, Fredholm, Herglotz, Bureau, Kodaira, etc.). In this paper an exhaustive theory is given for the first time for the most general equation with analytic coefficients. The author's method can be extended also to linear elliptic systems with analytic coefficients.

Let $L[u]=0$ be the equation in question. One can find an adjoint operator $\bar{L}[v]$ (which is kept fixed in what follows) such that for a domain R with sufficiently smooth boundary B and for any two functions $u(x)=u(x_1, \cdots, x_n)$, $v(x)=v(x_1, \cdots, x_n)$ of class C^m,

$$\int_R\{vL(u)-u\bar{L}(v)\}dx=\int_B M(u, v)dS,$$

where M is a bilinear expression involving derivatives of u and v of order not exceeding $m-1$ and depending linearly on the direction cosines of the normal to B. A function $K(z, x)$ which satisfies $L[u]=0$ with respect to x for $x\neq z$ and is such that for every v of class C^m and for z in B

$$(1) \qquad v(z)=\int_R K(z, x)\bar{L}[v]dx+\int_B M[K(z, x), v]dS_x$$

is called a fundamental solution of $L(u)=0$ with pole at z. It turns out that $K(z, x)$ is determined except for a regular function. It is analytic for $z\neq x$ in some neighborhood of z and behaves like Ar^{m-n} (r=distance between x and z) if n is odd, like $A \log r+Br^{m-n}$ if n is even. In the latter case the logarithmic term is dominant if $m\geqq n$, is either not present or not dominant if $m<n$. Analyticity of the fundamental solution and (1) imply the analyticity of every function of class C^m which satisfies a linear elliptic differential equation of order m with analytic coefficients and analytic right hand side.

A function u satisfying $L[u]=0$ except at one point, say the origin, is said to have there a pole of order s if all derivatives of u become infinite of order not greater than s at the origin, and s is the smallest integer having this property. There are no poles of positive order less than $n-1$. (This statement is a generalization of Riemann's theorem on removable singularities.) If $s=n-1$, u differs from a constant multiple of a fundamental solution by a regular solution. If $s>n-1$, then u is the sum of a regular solution and a linear combination with constant coefficients of the fundamental solution and its derivatives. The fundamental solution is constructed explicitly by means of particular solutions of $L[u]=0$ defined by Cauchy data on hyperplanes. The existence of these particular solutions follows from the

Cauchy-Kowalewski theorem. If the coefficients of the mth derivatives in $L[u]$ are constant, and all other coefficients are entire functions of the independent variables, the fundamental solution is regular at all points distinct from the pole. If all coefficients are constant, the fundamental solution can be obtained by quadratures. *L. Bers.*

Keldyš, M. V. On certain cases of degeneration of equations of elliptic type on the boundary of a domain. Doklady Akad. Nauk SSSR (N.S.) **77**, 181–183 (1951). (Russian)

Let $a(x, y)$, $b(x, y)$, $c(x, y)$ be analytic functions of their real variables, $c\leqq 0$. Set $L[u]=y^m u_{yy}+u_{xx}+au_y+bu_x+cu$. Let Δ be a simply connected domain bounded by the segment $(0, 1)$ of the x-axis and a smooth curve Γ situated in the upper half-plane. Two problems are considered for the equation (1) $L(u)=0$ in Δ: (D) u must assume given continuous values on the whole boundary of Δ; (E) u must assume given continuous values on Γ and be bounded in Δ. The following theorem is proved. (D) is uniquely solvable if $m<1$, or $m=1$ and $a(x, 0)<1$, or $1<m<2$ and $a(x, 0)\leqq 0$, or $m\geqq 2$ and $a(x, 0)<0$. (E) is uniquely solvable if $m=1$, $a(x, 0)\geqq 1$, or $1<m<2$ and $a(x, 0)>0$, or $m\geqq 2$ and $a(x, 0)\geqq 0$.
L. Bers (Los Angeles, Calif.).

Topolyanskiĭ, D. B. On the estimation of the generalized integral of Dirichlet in the plane problem of the theory of elasticity and in the three-dimensional boundary problem. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 423–428 (1950). (Russian)

The author shows how to obtain upper and lower bounds for the following two quadratic integrals: First,

$$\int_G\left\{a\left(\frac{\partial\varphi_1}{\partial x}-\frac{\partial\varphi_2}{\partial y}\right)^2+a\left(\frac{\partial\varphi_1}{\partial y}+\frac{\partial\varphi_2}{\partial x}\right)^2+b\left(\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_2}{\partial y}\right)^2\right\}dxdy,$$

where (φ_1, φ_2) is the solution of the "first boundary value problem of plane elasticity", i.e.,

$$a\Delta\varphi_1+b\left(\frac{\partial^2\varphi_1}{\partial x^2}+\frac{\partial^2\varphi_2}{\partial x\partial y}\right)=0,\quad a\Delta\varphi_2+b\left(\frac{\partial^2\varphi_1}{\partial x\partial y}+\frac{\partial^2\varphi_2}{\partial y^2}\right)=0$$

(where a and b are given positive elastic constants and $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$) in a plane domain G, and φ_1 and φ_2 assume prescribed values on the boundary of G. Secondly,

$$\int_G p(u_x^2+u_y^2+u_z^2)dxdydz + \int_G(2auu_x+2buu_y+2cuu_z+qu^2)dxdydz,$$

where u is the solution of a certain three-dimensional Dirichlet type problem. In both cases, an upper bound is obtained by the Rayleigh-Ritz method (i.e., Dirichlet's principle) and a lower bound is obtained by following the procedure given by E. Trefftz [Proc. 2d Internat. Cong. Applied Mech., Zürich, 1926, pp. 131–137 (1927)]. Reviewer's remark: For a large class of quadratic functionals, which includes the two particular ones considered here, upper and lower bounds may be obtained readily, in a systematic, unified, manner, by employing Schwarz' and Bessel's inequalities directly [Diaz and Weinstein, J. Math. Physics **26**, 133–136 (1947); these Rev. **9**, 211; Diaz, pp. 279–289 of Proc. Symposium on Spectral Theory and Differential Problems, Oklahoma A. and M. College, Stillwater, Okla., 1951]. *J. B. Diaz* (College Park, Md.).

Tanimoto, Bennosuke. Potentials of the simultaneous equations

$$(\nabla^2-aD)u_i=b\frac{\partial\theta}{\partial x_i}.$$

Bull. Earthquake Res. Inst. Tokyo **26**, 5–10 (1948).

Volkov, D. M. Bilinear integrals of linear hyperbolic problems. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 75–90 (1951). (Russian)

Let D be a domain in the Euclidean m-space bounded by a sufficiently smooth surface S. On S there is given a linear homogeneous boundary condition $\sigma u=0$ independent of time. Let u and v be solutions of the wave equation, $u=u(x_1, \cdots, x_m, t)$, $v=v(x_1, \cdots, x_m, t)$, defined and of class C^{2N+1} for $(x_1, \cdots, x_m)\epsilon D+S$, and for some given time interval, and satisfying the boundary condition. Let $\omega_N(u, v)$ be a bilinear homogeneous differential expression involving derivatives of u and v with respect to $x_1, \cdots, x_m, t$ up to the order N, with sufficiently smooth coefficients which are functions of $x_1, \cdots, x_m$ only. Let $\omega_{N-1}(u, v)$ be an expression of the same form defined on S. If

$$\text{(1)}\qquad I_N(u, v)=\int_D\omega_N(u, v)dx_1\cdots dx_m+\int_S\omega_{N-1}(u, v)dS$$

vanishes identically for all pairs u, v satisfying the conditions stated above, then I_N is called a bilinear integral of finite order (and of class 0) of the boundary value problem considered. Answering a question posed by Sobolev [C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 542–545 (1945); these Rev. **8**, 78] the author derives conditions which permit one to decide by a finite number of differentiations, eliminations, etc., whether or not a given expression (1) is a bilinear integral. *L. Bers* (Los Angeles, Calif.).

Weber, Maria. The fundamental solution of a degenerate partial differential equation of parabolic type. Trans. Amer. Math. Soc. **71**, 24–37 (1951).

Consider the Fokker-Planck equation

$$\text{(1)}\qquad \sum_{i,j=1}^n a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^n\left(x_i\frac{\partial u}{\partial y_i}+a_i\frac{\partial u}{\partial x_i}\right)+au+\frac{\partial u}{\partial t}=0 \qquad (a_{ij}=a_{ji}),$$

in which a_{ij}, a_i, a are functions of x_i, y_i, t, and $\sum a_{ij}\xi_i\xi_j$ is a positive definite form. As an equation of parabolic type it is degenerate since the second derivatives of u with respect to the y_i do not appear in the equation. With the coefficients of (1) satisfying certain differentiability and Lipschitz conditions in the region $-\infty<x_i$, $y_i<\infty$, $t_0\leqq t\leqq t_1$ the author constructs a fundamental solution for this equation (that is, a function with properties similar to the function $(y-\eta)^{-\frac{1}{2}}\exp(-(x-\xi)^2/4(y-\eta))$ termed the fundamental solution of $u_{xx}=u_y$). The method employed by the author follows the lines of E. E. Levi [Rend. Circ. Mat. Palermo **24**, 275–317 (1907)] for the construction of a fundamental solution for the elliptic case and extended by Feller [Math. Ann. **113**, 113–160 (1936)] and the reviewer [Duke Math. J. **13**, 61–70 (1946); these Rev. **7**, 450] to the nonsingular parabolic equation. *F. G. Dressel* (Durham, N. C.).

Storm, M. L. Heat conduction in simple metals. J. Appl. Phys. **22**, 940–951 (1951).

For the nonlinear parabolic equation

$$\text{(1)}\qquad \frac{\partial}{\partial x}\left[K\frac{\partial T}{\partial x}\right]=S\left(\frac{\partial T}{\partial t}\right)$$

in which the thermal parameters K and S are assumed to be functions of the dependent variable T, the author shows that if the expression

(2) $$(KdS/dT-SdK/dT)/(SK)^{\frac{1}{2}}$$

is constant then equation (1) can be transformed to a linear parabolic equation. The major portion of the paper is devoted to physical considerations. The object of these considerations is to justify that the expression in (2) is constant for the so-called "simple metals."

F. G. Dressel (Durham, N. C.).

Radok, J. R. M. Solution of a heat flow problem. Australian J. Sci. Research. Ser. A. 4, 12–15 (1951).

The following boundary value problem is considered. The function v is to be a solution of the heat equation $v_t=\alpha(v_{xx}+v_{yy})$ for $0<x<a$, $0<y<b$, $t>0$, and satisfy the following initial and boundary conditions:

$$v=0,\quad t=0;$$
$$v_x=0,\quad x=0;\quad v_x=c,\quad x=a;$$
$$v_y-\chi v=0,\quad y=0;\quad v_y+\chi v=0,\quad y=b.$$

Here α, c, χ are constants. The formula given for v does not satisfy all the desired conditions. *F. G. Dressel.*

***Sobolev, S. L. Uravneniya matematičeskoĭ fiziki. [The Equations of Mathematical Physics].** 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 424 pp.

This book reproduces an introductory course taught by the author at the Moscow State University. For the most part it deals with the classical equations of potentials, heat-conduction and wave-propagation. The theory of integral equations is developed and used, and so is Lebesgue integration. The presentation is rigorous, clear and vivid.

L. Bers (Los Angeles, Calif.).

***Levin, V. I., i Grosberg, Yu. I. Differencial'nye uravneniya matematičeskoĭ fiziki. [Differential Equations of Mathematical Physics].** Gosudarstv. Izdat. Tehn-Teor. Lit., Moscow-Leningrad, 1951. 575 pp.

A text-book intended for engineering students. The chapter headings are as follows. I. Statement of several fundamental problems of mathematical physics. II. Theory of the potential. III. The wave equation in an unbounded region; the method of characteristics. IV. Problems in characteristic functions. V. Solution of problems of mathematical physics by the method of characteristic functions. Appendix: Fundamental facts from the theory of cylinder functions.

Fichera, Gaetano. Sull'esistenza e sul calcolo delle soluzioni dei problemi al contorno, relativi all'equilibrio di un corpo elastico. Ann. Scuola Norm. Super. Pisa (3) 4, 35–99 = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. 248 (1950).

The present paper is concerned with showing the existence of and giving procedures for the calculation of the solutions of the boundary value problems of three-dimensional elasticity. The author has previously treated the boundary value problems of plane elasticity using analogous methods [Rend. Sem. Fac. Sci. Univ. Cagliari 18, 1–22 (1948); these Rev. 11, 700]. Let D be a bounded closed domain with a sufficiently smooth boundary FD. Analytically, the three classical boundary value problems consist in finding $u=(u_1, u_2, u_3)$, where the u_i are real valued functions (displacements) satisfying the system of partial differential equations

$$\Delta_2 u+k \operatorname{grad}\operatorname{div} u=f,\quad \text{on}\quad D-FD,$$

where $\Delta_2=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2+\partial^2/\partial x_3^2$, k is a given real number (elastic constant), $f=(f_1, f_2, f_3)$ is a given triple of functions, and u is subject to one of the three boundary conditions: (1) u prescribed on FD (displacements prescribed on the boundary); (2) $L(u)$ prescribed on FD (stresses prescribed on the boundary); (3) u prescribed on a subset F_1D of FD and $L(u)$ prescribed on $FD-F_1D$. (Here

$$L(u)=[(k-\lambda)\operatorname{div} u]\nu+(1+\lambda)du/d\nu+\lambda(\nu\wedge\operatorname{rot} u)$$

where λ is a given real number, ν is the inner normal of FD, and $\wedge$ denotes the vector product.) The method employed in this paper is based on the procedures introduced by M. Picone [Appunti di analisi superiore, Rondinella, Napoli, 1940, pp. 752–765] which rest upon a certain interpretation of Green's identities for the differential operator involved in a boundary value problem, and reduce the existence problem to the determination of certain "complete" sequences of functions. Green's identities are discussed in chapter I. Chapter II contains a generalized theory of potentials of surface distributions (containing the results of G. C. Evans and E. R. C. Miles [Amer. J. Math. 53, 493–516 (1931)] as special cases). Chapter III contains the required completeness theorems, which depend on a theorem of S. Banach [Théorie des opérations linéaires, Warsaw, 1932, p. 58, th. 7] and the generalized surface distribution results of chapter II. Chapters IV, V, and VI contain, respectively, the existence theorems for the boundary value problems with boundary conditions (1), (2), (3). The final chapter deals with procedures for the actual computation of the solutions. *J. B. Diaz* (College Park, Md.).

Amerio, Luigi. Sur le calcul des solutions des équations linéaires aux dérivées partielles de la technique. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. 288, 1–10 (1950).

Let $G_1, \cdots, G_p$ be p real-valued functions defined, respectively, on the subsets $T_1, \cdots, T_p$ of ν-dimensional Euclidean space, and write $G=(G_1, \cdots, G_p)$. If $\omega=(\omega_1, \cdots, \omega_p)$ is another "vector" (ordered p-tuple of functions as before), the scalar product (G, ω) is defined to be

$$(G, \omega)=\sum_{h=1}^{p}\int_{T_h} G_h\omega_h dT_h.$$

A "Fischer-Riesz system" of equations is a system

(*) $$\sum_{h=1}^{q} A_h c_{rh}+(G, \omega_r)=g_r,\qquad r=1, 2, \cdots,$$

where $\{\omega_r\}$ is a given sequence of vectors, $\{g_r\}$ is a given sequence of real numbers, the coefficients c_{rh}, $h=1, \cdots, q$, $r=1, 2, \cdots$, are given real numbers, while the vector G and the numbers $A_1, \cdots, A_q$ are unknown. M. Picone [Appunti di analisi superiore, Rondinella, Napoli, 1940, pp. 752–765; Ann. Sci. Univ. Jassy 26, 183–232 (1940); these Rev. 1, 236] has shown the equivalence (provided a suitable sequence $\{\omega_r\}$ can be determined) of certain boundary value problems for linear partial differential equations and systems (*). The author considers in this manner, making use of previous results [Ann. Mat. Pura Appl. (4) 24, 119–138 (1945); these Rev. 9, 94], the classical boundary value problem for thin elastic plates, which consists in integrating the equation

$$w_{xxxx}+2w_{xxyy}+w_{yyyy}=\Delta_4 w=f$$

in a plane domain D, subject to the following boundary

conditions on the boundary $\sigma=\sigma'+\sigma''+\sigma'''$ of D: (a) clamped edge along σ', (b) simple support along σ'', (c) free edge along σ'''. This boundary value problem is shown to be equivalent to a Fischer-Riesz system of type (*). In case σ''' is empty the corresponding system (*) simplifies, since then $c_{rh}=0$, $h=1, \cdots, q$; $r=1, 2, \cdots$. *J. B. Diaz.*

Difference Equations, Special Functional Equations

Myškis, A. D. On solutions of linear homogeneous differential equations of the second order of periodic type with a retarded argument. Mat. Sbornik N.S. 28(70), 15–54 (1951). (Russian)

The author considers the equation

$$(1)\qquad y''(x)+M(x)y(x-\Delta(x))=0\quad (M(x)\geqq 0, \Delta(x)\geqq 0).$$

The general properties of its solutions have been studied by the author [Uspehi Matem. Nauk (N.S.) 4, no. 5(33), 99–141 (1949); these Rev. 11, 365]. $M(x)$, $\Delta(x)$ are given for $A\leqq x<B$ $(-\infty<A<B\leqq\infty)$. The initial conditions are in terms of a number C' and a function $\varphi(x)$, defined for $-\infty<x\leqq A$. A solution $y(x)$ is supposed to be twice derivable for $A\leqq x<B$ $(A<B_1\leqq B)$; for $x-\Delta(x)<A$ one considers that $y(x-\Delta(x))\equiv\varphi(x-\Delta(x))$; $y(A)=\varphi(A)$, $y'(A)=C'$. Define $\gamma(x)$ for $-\infty<x<\infty$ as the upper bound of the numbers t, for which $t-\Delta(t)<x$; when there are no such t for $x\leqq A$, let $\gamma(x)=A$. Let $\Delta_0=\sup_{[A,B)}\Delta(x)$, $M_0=\sup_{[A,B)}M(x)$, $m_0=\inf_{[A,B)}M(x)$. The segment $[a_1, a_2]$ $(a_1<a_2)$ is the s.c. (semicycle) for $f(x)$ if $f(a_1)=f(a_2)=0$ and $f(x)\neq 0$ on the interval (a_1, a_2). If $f(x)>0$ (<0), the s.c. is "positive" ("negative"). With $a_1\geqq A$, the s.c. is "great", if $a_2>\gamma(a_1)$, "small" in the contrary case. Some of the typical results are as follows. Let (2) $\Delta_0<M_0^{-\frac{1}{2}}(\frac{1}{2}\pi+\sqrt{2})=v_0$; let a_1, a_2 be given with $\gamma(a_1)\leqq a_2<B_1$; a solution $y(x)\geqq 0$ $(\leqq 0)$ on $[a_1, a_2]$ is given such that $y(a_2)=0$, $y'(a_2)\leqq 0$ $(\geqq 0)$; then either $y(x)\equiv 0$ for $x>a_2$ or, for some a_2' on $[a_2, B)$, one has $y(x)\equiv 0$ for $a_2\leqq x\leqq a_2'$, and either $y(x)<0$ (>0) for $x>a_2'$, or a_2' is the left end-point of a "great" s.c. $[a_2', a_3)$ of the solution $y(x)$, while (3) $a_3-a_2'\geqq v_0$ and $y(x)<0$ (>0) on (a_2', a_3). If (2) holds, $m_0>0$ and $[a_1, a_2]$ is a great s.c. for $y(x)$, then: (A) if $B_1=\infty$, the semiaxis $[a_2, \infty)$ consists of an infinity of s.c.'s $[a_2, a_3]$, $[a_3, a_4]$, $\cdots$, on which the sign of $y(x)$ alternates, while (4) $a_{k+1}-a_k\geqq v_0$; (B) if $B_1<\infty$, then $[a_2, B_1)$ consists of a finite number of great s.c.'s $[a_2, a_3]$, $\cdots$, $[a_{k_0}, a_{k_0+1}]$ and the semi-segment $[a_{k_0+1}, B_1)$, on which the sign of $y(x)$ alternates, while (4) holds. A detailed and definitive study is given of the form of the solution on its great s.c.'s. Further, there is a comprehensive study of the possibility of occurrence of small s.c.'s. Finally, in the case when in (1) one has $M(x)=1$, the possibility of damping is studied.

W. J. Trjitzinsky (Urbana, Ill.).

Germay, R.-H. Application de la méthode des fonctions majorantes à des systèmes d'équations récurrentes définissant des fonctions implicites. Ann. Soc. Sci. Bruxelles. Sér. I. 65, 64–70 (1951).

Employing the method of majorants, the author establishes the existence of an analytic solution of a set of analytic recurrence equations. *R. Bellman.*

Thielman, H. P. A note on a functional equation. Amer. J. Math. 73, 482–484 (1951).

The reviewer [Bull. Soc. Math. France 76, 59–64 (1948); these Rev. 10, 685] has proved that: (1) Every continuous increasing and associative $[xo(yoz)=(xoy)oz]$ function xoy can be written in the form $xoy=\phi^{-1}[\phi(x)+\phi(y)]$ [the author quotes a weaker result postulating also commutativity, $xoy=yox$, which was proved to follow continuity, strict monotony, and associativity]; (2) if the functional equation $\phi(ax+by+c)=\alpha\phi(x)+\beta\phi(y)+\gamma$ has a nontrivial solution then $\alpha=a$, $\beta=b$, and the general solution $\phi(x)$ is a linear function [Comment. Math. Helv. 21, 247–252 (1948); these Rev. 9, 514]; (3) if the functional equation

$$\phi[F(x, y)]=G[\phi(x), \phi(y)],$$

where $G(x, y)=ax+by+c$, or more generally

$$G(x, y)=\psi^{-1}[a\psi(x)+b\psi(y)+c],$$

has a nontrivial solution, then $F(x, y)=\phi^{-1}[a\phi(x)+b\phi(y)+\gamma]$ [Acad. Serbe Sci. Publ. Inst. Math. 2, 257–262 (1948); these Rev. 10, 303]. The author proves that if the functional equation $\phi[xoy]=\varphi(x)+\phi(y)$, where xoy is a polynomial, has a continuous increasing solution, then $\exp\phi(x)$ must be a linear function, or what is the same, the only continuous increasing associative polynomial is $xoy=axy+b(x+y)+c$, where $ac=b^2-b$. The author's proof of this theorem is unnecessarily long, since after showing that associativity and commutativity imply that xoy is a symmetric polynomial of degree 1 in x and also in y, the substitution of $xoy=axy+bx+by+c$ into $(xox)oy=xo(xoy)$ immediately gives $ac=b^2-b$. The author also proves that the continuous increasing associative functions for which xox is a polynomial of degree greater than 1 are the $xoy=\phi^{-1}[\phi(x)+\phi(y)]$ formed with $\phi(x)=k\ln(ax+b)$ or with $\phi(x)=k\arccos(ax+b)$.

J. Aczél (Miskolc).

Integral Equations

Kreĭn, M. G. Determination of the density of a nonhomogeneous symmetric cord by its frequency spectrum. Doklady Akad. Nauk SSSR (N.S.) 76, 345–348 (1951). (Russian)

The integral equation $\phi(x)=\lambda\int_{-1}^{1}K(x, s)\phi(s)d\sigma(s)$ is considered where $K(s, x)=K(x, s)=\frac{1}{2}(1+x)(1-s)$ for $x\leqq s$. Here $\sigma(-1)=0$ and $\sigma(x)$ is monotone nondecreasing. The eigenvalues are denoted by $\lambda_k=p_k^2$. The first result is $\lim n/p_n=\pi^{-1}\int_{-1}^{1}[\sigma'(x)]^{\frac{1}{2}}dx$. In the symmetric case

$$\sigma(1)-\sigma(x+0)=\sigma(-x-0)-\sigma(-1)$$

the theorem is given that a necessary and sufficient condition that there exist a symmetric $\sigma(x)$ corresponding to a given $0<p_0<p_1\cdots$ is the absolute convergence of $\Delta(\lambda)=\prod(1-\lambda/\lambda_n)$, $\lambda_n=p_n^2$, and the inequality

$$-\sum 1/(\lambda^2_{2n}\Delta'(\lambda_{2n}))<\infty.$$

Other results are given. *N. Levinson.*

Nickel, Karl. Lösung eines Integralgleichungssystems aus der Tragflügeltheorie. Math. Z. 54, 81–96 (1951).

The author considers the integral equation

$$(1)\qquad \frac{1}{\pi}\sum_{\nu=1}^{n}\int_{a_\nu}^{b_\nu}\frac{f_\nu(y_\nu)}{x_\mu-y_\nu}dy_\nu=g_\mu(x_\mu),\qquad \mu=1, \cdots, n,$$

where $a_1\leqq x_1\leqq b_1<a_2\leqq\cdots<a_n\leqq x_n\leqq b_n$ and the Cauchy principal value is to be taken. It is assumed that the $g_\mu(x)$ are measurable for $a_\mu\leqq x\leqq b_\mu$ and that

$$\int_{a_\nu}^{b_\nu}|g_\nu(x)|^p[(b_\nu-x)(x-a_\nu)]^{\frac{1}{2}(p-1)}dx$$

exists for $\nu=1, \cdots, n$. The author shows that the most

general solution of (1) is

$$f_\nu(y_\nu)=-\frac{(-1)^\nu}{\pi}[-\prod_\kappa(b_\kappa-y_\nu)(a_\kappa-y_\nu)]^{-\frac{1}{2}}$$

$$\times\left\{\sum_{\mu=1}^{n}(-1)^\mu\int_{a_\mu}^{b_\mu}\frac{g_\mu(x)}{y_\nu-x}[-\prod_\lambda(b_\lambda-x)(a_\lambda-x)]^{\frac{1}{2}}dx+P_{n-1}(y_\nu)\right\},$$

$a_\nu\leqq y_\nu\leqq b_\nu$, $\nu=1,\cdots,n$, where

$$P_{n-1}(y)=A_{n-1}y^{n-1}+\cdots+A_1y+A_0$$

has arbitrary coefficients and $A_{n-1}=(-1)^{n-1}\sum_\nu\int_{a_\nu}^{b_\nu}f_\nu(x)dx$. The proof is by induction on n, the proof for $n=1$ having been established in an earlier paper [Math. Z. **53**, 21–52 (1950); these Rev. **12**, 423]. The author's statement that only the cases $n=1$ and 2 had been considered previously overlooks the fact that similar theorems can be found in the work of Mushelišvili [e.g., Singular integral equations . . . , OGIZ, Moscow-Leningrad, 1946, especially §88; these Rev. **8**, 586] or of Mihlin [Integral equations . . . , 2d ed., OGIZ, Moscow-Leningrad, 1949, §27; these Rev. **12**, 712].

J. V. Wehausen (Providence, R. I.).

Dörr, J. Strenge Lösung der Integralgleichung für die Strömung durch ein senkrechtes Flügelgitter. Ing.-Arch. **19**, 66–68 (1951).

Solution of the integral equation

$$f(x)=(1/2h)\int_{-1}^{1}g(y)\coth\pi(x-y)h^{-1}dy$$

where $h=h'+ih''$ by reducing it essentially by a suitable transformation to $f^*(x^*)=\int_{-1}^{1}(y^*-x^*)^{-1}g^*(y^*)dy^*$. (Cauchy principal values are to be taken in the integrals.) The result appears to be related to earlier work by Pistolesi and others.

E. Reissner (Cambridge, Mass.).

Parodi, Maurice. Quelques applications de calcul symbolique à deux variables à la résolution d'équations intégrales. Ann. Soc. Sci. Bruxelles. Sér. I. **65**, 57–63 (1951).

The author points out that the two-dimensional Laplace transformation can be used to solve certain integral equations, and illustrates his remarks by some examples.

A. Erdélyi (Pasadena, Calif.).

Functional Analysis, Ergodic Theory

Menger, Karl. Espaces vectoriels généraux, topologies triangulaires, transformations linéaires généralisées. C. R. Acad. Sci. Paris **232**, 2176–2178 (1951).

A generalized norm is a real-valued functional satisfying only the condition of positive homogeneity in the most general case. After a review of results on vector spaces with generalized norms [Canadian J. Math. **1**, 94–104 (1949); Revista Ci., Lima **50**, 155–165 (1948); these Rev. **10**, 306, 549] generalized linear transformations of such spaces are defined and their structure is determined in the two-dimensional "hyperbolic" case.

D. H. Hyers.

Kowalsky, Hans-Joachim. Differenzenquotienten in lokalkonvexen Vektorräumen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1950**, 13–32 (1951).

The space considered by the author is a topological vector space R over a topological field K of characteristic zero. No separation or completeness axioms are postulated for R or K, but it is assumed that the topology of K is such that when the rational field K_0 is considered as a subspace of K, the relative topology of K_0 reduces to the ordinary topology given by the absolute value. The author calls a set in R convex if it contains with each pair x, y the set of all vectors $ax+by$, where a, b are positive rational numbers whose sum is one. The condition of local convexity, needed for integration, means of course that there exists a complete neighborhood system of the origin consisting entirely of convex sets.

For functions $x(\tau)$ on an open set of K to R, generalizations of some results of E. Hopf [Dissertation, Berlin, 1925] are given concerning the existence of nth derivatives vs. that of limits of nth difference quotients. The main analytical tool used is the Taylor series with remainder in the form of an integral of Riemann type. Although explicitly stated that no separation axiom is required, nothing is said regarding the difficulties which may result due to multiple valued limits. This throws doubt on both the meaning and proof of the author's version of the fundamental theorem of integral calculus (p. 22) and on all the results that follow. No mention is made of related and much earlier results of various authors on abstract differential calculus.

D. H. Hyers (Los Angeles, Calif.).

Edwards, R. E. The translations of a function holomorphic in a half-plane and related problems in the real domain. Proc. London Math. Soc. (3) **1**, 118–128 (1951).

Consider functions $f(\lambda)$ $(\lambda=\sigma+i\tau)$ defined and holomorphic when $\sigma>0$. Suppose that, for each $a>0$,

$$p_a(f)=\sup_{\sigma>a}\left\{\int_{-\infty}^{\infty}|f(\sigma+i\tau)|^2d\tau\right\}^{\frac{1}{2}}<\infty.$$

Denote by E the class of f satisfying this condition. Then E is a locally convex but nonnormable topological linear space with the sets $\{f\epsilon E|p_a(f)<\epsilon\}$ (a, ϵ independent positive parameters) as a basis of neighborhoods of the origin. There is a close relation between E and the class of functions $\phi(x)$ defined almost everywhere (a.e.) on $(0,\infty)$ and such that $e^{-2\pi\sigma x}\phi(x)\epsilon L^2(0,\infty)$ for every $\sigma>0$. Following Paley and Wiener and using Plancherel's theorem, one may show that, corresponding to $f\epsilon E$, there is a ϕ of the foregoing class such that

$$p_\sigma(f)=\left\{\int_0^\infty|e^{-2\pi\sigma x}\phi(x)|^2dx\right\}^{\frac{1}{2}}\qquad(\sigma>0).$$

There is then a correspondence $\phi=Tf$; the function ϕ is defined uniquely save perhaps on a null set, and it uniquely determines f. The set of points at which $\phi(x)=0$ is denoted by Z_f. The author shows that the general representation of a linear functional on E is

$$F(f)=\int_0^\infty\phi(x)u(x)dx,\qquad\phi=Tf,$$

where $u(x)$ is defined a.e. on $(0,\infty)$ and such that $e^{2\pi ax}u(x)\epsilon L^2(0,\infty)$ for some $a>0$. The class of such $u(x)$ may be identified with E', the topological dual space of E.

Let A be a set of points α with $\Re(\alpha)\geqq0$, and consider translations $f_\alpha(\lambda)=f(\lambda+\alpha)$, where $f\epsilon E$. Let Af denote the closed linear manifold of E spanned by the f_α, and let Df denote the closed linear manifold spanned by the derivatives $f^{(n)}$, $n=0,1,2,\cdots$. Finally, for any real c, let L_c be the class of functions representable in the form $\int_0^\infty e^{-2\pi\lambda x}m(x)dx$ with $m(x)$ subject to the condition $\int_0^\infty e^{2\pi cx}|m(x)|dx<\infty$. The author's first main theorem, and the only one which we shall quote, is as follows: Let A be such that, for every $c>0$, if a

function in L_0 vanishes on A it vanishes identically. Then Af and Df coincide and consist of those $g\epsilon E$ for which $Z_g \supset Z_f$ modulo null sets. In particular, $Af=Df=E$ if and only if Z_f is a null set. The proof, which is simple, relies mainly on the foregoing representation of linear functionals. There is a dual theorem, dealing with closed linear manifolds in E', and a theorem of like character for a certain subspace of E which is a Hilbert space. *A. E. Taylor.*

Rubinštein, G. Š. On the isolation and separation of convex sets by hyperplanes. Doklady Akad. Nauk SSSR (N.S.) 78, 213–215 (1951). (Russian)

Let E be an arbitrary real linear vector space. A point x of a subset M in E is called a (*)-interior point of M if it is an interior point of the intersection with M by an arbitrary straight line through x. If every point of M is (*)-interior, then M is said to be (*)-open. The (*)-open sets determine in E a (*)-topology. A convex set which contains at least one (*)-interior point is called a convex body. The author proves the following two theorems which generalize known results for normed linear spaces [Mazur, Studia Math. 4, 70–84 (1933); Eidelheit, ibid. 6, 104–111 (1936)]. Theorem A: If a linear manifold L does not contain a (*)-interior point of a convex body M, then there exists a hyperplane H containing L such that M is contained in one of the half-spaces determined by H. Theorem B: If a convex set M does not contain a (*)-interior point of a convex body N, then there exists a hyperplane H which separates M and N. *C. E. Rickart* (New Haven, Conn.).

Sebastião e Silva, J. Integration and derivation in Banach spaces. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) 1, 117–166 (1950); errata 401–402 (1951). (Portuguese)

This is an expository paper. The topics covered are: line integrals, Fréchet differentials (viewed as derivatives), multilinear operators, Taylor's formula with remainder, implicit functions, exact differentials, differential equations. The author has drawn heavily on the work of T. H. Hildebrandt and L. M. Graves [Trans. Amer. Math. Soc. 29, 127–153 (1927); Graves, ibid., 163–177, 514–552 (1927)]. To some extent he duplicates work of M. Kerner [Ann. of Math. (2) 34, 546–572 (1933)]. The latter paper was called to the author's attention by the reviewer after completion of the present paper.

The line integral $\int_\Gamma f(x)dx$ is defined as an element of a Banach space Y. The rectifiable curve Γ $(x=x(t),\ t_0\leqq t\leqq t_1)$ is in a Banach space X, and f is a function defined and continuous on Γ, with values in the space $[X, Y]$ (the reviewer's notation) of bounded linear operators mapping X into Y. Some inaccuracies in connection with the discussion of the integral are corrected in a subsequent note in the same journal. Following Michal and Zorn, the author deals with derivatives instead of with Fréchet differentials. If $F(x)$ maps a part of X into Y, $F'(x)$ is defined with values in $[X, Y]$, $F'(x)[\delta x]$ being then the Fréchet differential $dF(x;\delta x)$. One then has $\int_\Gamma F'(x)dx = F(x(t_1))-F(x(t_0))$. The systematic use of derivatives rather than differentials is the essential novelty of the paper. The errata contain corrections of minor inaccuracies and misprints and two additional titles for the bibliography. *A. E. Taylor.*

Melvin-Melvin, H. On generalized K-transformations in Banach spaces. Proc. London Math. Soc. (2) 53, 83–108 (1951).

The results of this paper concern generalizations of the Toeplitz-Kojima-Schur and Hahn theorems on linear methods of summation to matrices of linear operators and are essentially equivalent to theorems III, IV, V of A. Robinson [same Proc., (2) 52, 132–160 (1950); these Rev. 12, 253] except that they are formulated for the special case of a family of operators A such that $\|A_nx-Ax\|\to 0$ for all x implies $\|A_n-A\|\to 0$. *G. G. Lorentz* (Toronto, Ont.).

Taylor, A. E. Banach spaces of functions analytic in the unit circle. I. Studia Math. 11, 145–170 (1950).

In addition to the usual axioms for a complex normed linear vector space of analytic functions on the unit circle the author imposes additional ones (see $P_1, \cdots, P_4$ below) and develops an abstract theory which is applicable to a number of important classes of analytic functions. These classes include the spaces H^p, $1\leqq p\leqq\infty$, and the space of functions analytic for $|z|<1$ and continuous on the closed unit circle. Specific applications to these spaces are reserved for another publication and the present paper is devoted to the general theory.

Let B be a complex normed linear vector space each element of which belongs to the class $\mathfrak{A}$ of functions single valued and analytic in the open set of complex numbers $\Delta=[z;|z|\leqq 1]$. Let γ_n, $n=0,1,\cdots$, be the linear functional defined by $\gamma_n(f)=f^{(n)}(0)/n!$ and let u_n be the element of $\mathfrak{A}$ defined by $u_n(z)=z^n$. For real x and complex w with $|w|\leqq 1$ the distributive operators U_xf, T_wf in $\mathfrak{A}$ are defined by $f(ze^{ix})$, $f(zw)$ respectively. The space B is said to be a space of type $\mathfrak{A}_k$, $k=1,2,3,4$, in case it satisfies the axioms $P_1,\cdots,P_k$ which follow. (P_1) The supremum of the norm $|\gamma_n|$ for $n=0,1,2,\cdots$ is a finite constant $A_1(B)$. (P_2) The functions u_n, $n=0,1,2,\cdots$, are in B and $\sup_n|u_n|=A_2(B)$ is finite. (P_3) For x real U_x is a unitary map of B into B. (P_4) For $0\leqq r<1$, T_r maps B into B and $\sup_{0\leqq r<1}\|T_r\|=A_4(B)$ is finite. Concerning spaces B of type $\mathfrak{A}_1$ the author proves, among other results, the following two theorems. Theorem. Let f_n, $f\epsilon B$, $\sup\|f_n\|<\infty$, and $\gamma_k(f_n)\to\gamma_k(f)$, $k=0,1,2,\cdots$. Then $f_n(z)$ converges to $f(z)$ uniformly on compact subsets of Δ. Theorem. Let B_1 and B_2 be complete and suppose that each element of B_1 is also in B_2. Then the identity mapping of B_1 into B_2 is continuous. Concerning spaces B of type $\mathfrak{A}_2$ the following results are proved. Theorem. If $f(z)$ is analytic for $|z|<R$ with $R>1$ then $f\epsilon B$. Theorem. If f, $f_n\epsilon\mathfrak{A}$ and $f_n(z)\to f(z)$ uniformly on compact subsets of Δ then $\|T_wf_n-T_wf\|\to 0$ uniformly in w on compact subsets of Δ. Theorem. If f is an element of B such that its Maclaurin series converges in the norm of B then if w is restricted to an angular region (of angle less than π), between two chords of the unit circle which meet at $w=1$ we have $\|T_wf-f\|\to 0$ as $w\to 1$. Concerning spaces of type $\mathfrak{A}_3$ it is shown that $\|T_w\|=\|T_r\|$ if $|w|=r<1$, that $\|T_r\|$ is non-decreasing and continuous in r, and that $\log\|T_r\|$ is a convex function of $\log r$. Certain spaces related to the conjugate spaces are discussed. We shall mention only one of the results. If f, $g\epsilon\mathfrak{A}$ have the developments $f(z)=\sum a_nz^n$, $g(z)=\sum b_nz^n$, define the bilinear form $B(f,g;z)=\sum a_nb_nz^n$, $z\epsilon\Delta$. For a space B define B^0 as the class of all $F\epsilon\mathfrak{A}$ such that the $\lim_{r\to 1}B(f,F,r)$ exists for each $f\epsilon B$. Theorem. Let B be a Banach space of type $\mathfrak{A}_4$ such that T_rf converges weakly to f as $r\to 1$, for each $f\epsilon B$. Then every linear functional $\gamma\epsilon B^*$ is representable in the form

$$\gamma(f)=\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}f(\rho e^{i\theta})F\left(\frac{r}{\rho}e^{-i\theta}\right)d\theta,\quad f\epsilon B,$$

where $r<\rho<1$ and $F\epsilon B^0$. The element F of B^0 uniquely determines and is uniquely determined by γ. Furthermore,

if we define

$$\|F\|'=\lim_{r\to 1}\sup_{\|f\|=1}|B(f,F,r)|,\quad\text{then}\quad \|\gamma\|\leqq\|F\|'\leqq A_4(B)\|\gamma\|.$$

N. Dunford (New Haven, Conn.).

Hoheisel, Guido. Über Alternativsätze und Vielfachheit der Eigenwerte. Math. Nachr. **5**, 231–236 (1951).

This note gives a proof of the fact that if K is a linear completely continuous transformation on a Hilbert space into itself, then the number of linearly independent characteristic functions of K corresponding to λ_0, i.e. solutions of the homogeneous equation $x-\lambda_0K(x)=0$, is the same as that of the adjoint K^* corresponding to $\bar{\lambda}_0$, the method of proof being the extension to Hilbert space of that given in the author's book on integral equations [Integralgleichungen, de Gruyter, Berlin-Leipzig, 1936, Kap. 1, §4]. How to alter the proof so as to make it available for the general linear normed complete space [see e.g. Zaanen, Nieuw Arch. Wiskunde (2) **22**, 269–282 (1948); these Rev. **9**, 448] is not indicated. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Gohberg, I. On linear equations in Hilbert space. Doklady Akad. Nauk SSSR (N.S.) **76**, 9–12 (1951). (Russian)

This paper extends previous results of Nikol'skiĭ [Izvestiya Akad. Nauk SSSR. Ser. Mat. **7**, 147–166 (1943); these Rev. **5**, 187] and Mihlin [Uspehi Matem. Nauk (N.S.) **3**, no. 3(25), 29–112 (1948); these Rev. **10**, 305] on the formulation in abstract spaces of certain results of integral equations. It is concerned particularly with finding necessary and sufficient conditions that a bounded linear operator A in a Hilbert space H satisfy the following conditions: α) the equation $A\varphi=0$ has a finite number of linearly independent solutions; β) the operator A is normally soluble (i.e. the set $\{A\varphi;\,\varphi\epsilon H\}$ is the orthogonal complement of the set $\{\psi;\,A^*\psi=0\}$, A^* the adjoint of A). *J. V. Wehausen.*

Gohberg, I. C. On linear equations in normed spaces. Doklady Akad. Nauk SSSR (N.S.) **76**, 477–480 (1951). (Russian)

The author generalizes a theorem of Nikol'skiĭ [reference cited in preceding review]. Let A be a bounded linear operator on a complex normed linear space R, and $\bar{A}$ the adjoint operator on $\bar{R}$. Then the following statements are equivalent: (1) The equations $Ax=0$ and $\bar{A}X=0$ have a finite number of linearly independent solutions, $\alpha(A)$ and $\beta(A)$ respectively, A is normally soluble (see the preceding review for definition, with some necessary rephrasing for a normed space), and $\alpha(A)-\beta(A)=\kappa(A)\leqq 0$ $(\geqq 0)$. (2) A can be represented as $A=D+T$ where D has a left (right) inverse, $\bar{D}X=0$ $(Dx=0)$ has exactly $-\kappa$ (κ) linearly independent solutions, and T is completely continuous. (3) $A=D+K$ with D as in (2) and $K(R)$ finite-dimensional. (4) $\bar{A}=D^*+K^*$, where D^* has a right (left) inverse, $D^*X=0$ $(\bar{D}^*\Psi=0)$ has $-\kappa$ (κ) linearly independent solutions, and $K^*(\bar{R})$ is finite-dimensional. (5) $\bar{A}=D^*+T^*$, D^* as in (4) and T^* completely continuous. This generalization of Nikol'skiĭ's theorem is sufficiently general to apply to certain classes of singular integral equations [cf. F. Noether, Math. Ann. **82**, 42–63 (1921)]. *J. V. Wehausen* (Providence, R. I.).

Atkinson, F. V. The normal solubility of linear equations in normed spaces. Mat. Sbornik N.S. **28(70)**, 3–14 (1951). (Russian)

The purpose of this paper is similar to that of the paper reviewed above, namely, to generalize a theorem of Nikol'skiĭ [see the preceding review for references and notation] so as to relax the requirement that $\alpha(A)=\beta(A)$ and to include certain singular integral equations. Theorem 1 is a list of equivalents similar to that in the paper reviewed above with the addition of the following: There exists a bounded linear operator U such that $AU=I-K_1$ and $UA=I-K_2$, I the identity, $K_i(R)$ finite-dimensional. The author goes on to study the function $\kappa(A)$ where A satisfies the conditions of this theorem. Such operators are called "generalized Fredholm operators" and the set of them is denoted by S. Theorem 2. If $A_1,\,A_2\epsilon S$, then $A_1A_2\epsilon S$ and $\kappa(A_1A_2)=\kappa(A_1)+\kappa(A_2)$. Theorem 3. If $A\epsilon S$ and V is completely continuous, then $A+V\epsilon S$ and $\kappa(A+V)=\kappa(A)$. Theorem 4. If $A\epsilon S$, there exists $p>0$ such that if $\|A'-A\|<p$ then $A'\epsilon S$ and $\kappa(A')=\kappa(A)$. In the last part of the paper the author applies the theory of normed rings to formulate abstractly a theorem of Mihlin on singular integral operators. This is done by introducing a certain normed ring of operators, S_2, and taking the factor ring S_2/Z, where Z is the closed ideal of completely continuous operators. If $T\epsilon S_2$ and T' is the corresponding element in S_2/Z, then $T\epsilon S$ if T' has an inverse; T' has an inverse if a certain series $\sum_{-\infty}^{\infty}a_nz^n$ does not vanish for z in a certain region Γ. This tying-up of T with a region of the complex plane and a complex function is the generalization of Mihlin's theorem [Mat. Sbornik N.S. **3(45)**, 121–141 (1938)]. *J. V. Wehausen.*

Gohberg, I. C. On linear operators depending analytically on a parameter. Doklady Akad. Nauk SSSR (N.S.) **78**, 629–632 (1951). (Russian)

The first part of this paper generalizes results in the second part of the paper by Nikol'skiĭ cited in the third preceding review. Let A_λ be a linear operator on a normed linear space R with A_λ a holomorphic function of λ for λ in some region of the complex plane. The "Fredholm region" for A_λ is the set Φ_{A_λ} of the values λ for which $T_\lambda=I-A_\lambda$ can be written $T_\lambda=D+T$ (or $D+K$), where D has an inverse, T is completely continuous and $K(R)$ is finite-dimensional. Then Φ_{A_λ} is open and can be decomposed into components: $\Phi_{A_\lambda}=\sum_{i=1}^{\infty}\Phi_i$. Theorem 1. In each component Φ_i there exists an isolated set F_i such that for $\lambda\epsilon\Phi_i-F_i$ the equation $(E-A_\lambda)x=0$ has the same number of solutions; for $\lambda\epsilon F_i$, the equation has more solutions.

Consider now an operator $A_\lambda=E-\lambda U$, U a bounded operator on R. The "Noether region" for U is the set N_U of values λ for which $A_\lambda=D+T$, where T is completely continuous and either D has a left inverse and $\bar{D}X=0$ has a finite number of solutions, or D has a right inverse and $Dx=0$ has a finite number of solutions. Such operators have been characterized in each of the two papers reviewed above. The set N_U may be decomposed into components: $N_U=\sum_{i=1}^{\infty}N_i$. Theorem 2. The index $\kappa(A_\lambda)$ [see the second preceding review for definition] is constant on each component N_i. Theorem 3. In each N_i there is an isolated set M_i such that $A_\lambda x=0$ has the same number of solutions for each $\lambda\epsilon N_i-M_i$, but has more solutions than this for $\lambda\epsilon M_i$. *J. V. Wehausen* (Providence, R. I.).

Pini, Bruno. Spazio duale dello spazio delle matrici infinite limitate. Ann. Mat. Pura Appl. (4) **31**, 111–128 (1950).

The paper concerns necessary and sufficient conditions on the infinite matrix $A=\{a_{hk}\}$ so that $\sum_{h,k}a_{hk}x_{hk}$ be convergent for every matrix $X=\{x_{hk}\}$ (a) limited in the Hilbert sense, (b) completely continuous. A necessary condition is that

$\sum a^2_{hk}$ be convergent, sufficient that there exist matrices b_{hk}, c_{hk} such that $\sum b^2_{hk}$ and $\sum c^2_{hk}$ be convergent and $a_{hk}=\sum_i b_{hi}c_{ik}$. Necessary and sufficient condition for both (a) and (b) is that there exist a constant M such that for all x_{hk}, m and n, $|\sum_{h=1}^m \sum_{k=1}^n a_{hk}x_{hk}| \leqq M\cdot M_X$ where M_X is the modulus in the Hilbert sense of the matrix x_{hk}, $h=1, \cdots, m$; $k=1, \cdots, n$. Hence the adjoint spaces of (a) and (b) coincide. Additional properties are deduced for the case where the matrix A is symmetric. Some of the proofs could have been simplified by the use of well known theorems on linear forms in linear normed complete spaces. *T. H. Hildebrandt.*

Magenes, Enrico. Sulle estremanti dei polinomiali nella sfera di Hilbert. Rend. Sem. Mat. Univ. Padova **20**, 24–47 (1951).

Let A be a measurable set in the p-dimensional Euclidean space S_p, let L^2 be the Hilbert space of all real $f(t)$ for which $\int_A f(t)^2 dt < \infty$, and let, for $i=1, 2, \cdots, n$, $K_i(t_1, t_2, \cdots, t_i)$ be real functions for which

$$\int_A \cdots \int_A K_i(t_1, t_2, \cdots, t_i)^2 dt_1 dt_2 \cdots dt_i < \infty.$$

We then consider for $f \subset L^2$ the "integral form of degree i"

$$(1)\quad P_i(f) = \int_A \cdots \int_A K_i(t_1, t_2, \cdots, t_i) \times f(t_1)f(t_2)\cdots f(t_i)dt_1dt_2\cdots dt_i$$

and the "polynomial of degree n" (2) $P(f)=\sum_{i=1}^n P_i(f)$. Let Γ_δ be the solid sphere of radius δ of L^2 whose center is the zero element of L^2, and $F\Gamma_\delta$ its boundary. It is well known that $P(f)$ reaches a maximum and a minimum in Γ_δ. Moreover, according to a theorem of Fichera [Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 174–177 (1947); these Rev. **9**, 43] the maximum and minimum values of $P(f)$ in Γ_δ are at the same time the least upper and the greatest lower bounds respectively of $P(f)$ on $F\Gamma_\delta$. By results of Tonelli [Ann. Mat. Pura Appl. (4) **18**, 1–21 (1939); these Rev. **1**, 77], Picone [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **29**, 155–159 (1939)], and Fichera [Ist. Naz. Appl. Calcolo (2) no. **160** (1944); these Rev. **8**, 278, 708], it has been established that at least one of the two extreme values (maximum or minimum) is always taken on $F\Gamma_\delta$ in each of the following two cases: a) $n=2$; b) $P=P_n$, i.e. P is a homogeneous form of degree n. The present paper deals with a general polynomial $P(f)$ of the form (2). The author shows by an example that the above result for the cases a) and b) is not true in the general case. The main part of the paper deals with the statement and proof of various sufficient conditions for the validity of the result in question. An application to the theory of nonlinear integral equations concludes the paper. *E. H. Rothe.*

Krasnosel'skiĭ, M. A. On certain types of extensions of Hermitian operators. Ukrain. Mat. Žurnal **2**, no. 2, 74–83 (1950). (Russian)

In this article an operator A in a unitary space $\mathfrak{H}$ is called hermitian if $(Af, g)=(f, Ag)$ for all f and g in its domain, $\mathfrak{D}(A)$, irrespective of whether its domain is dense in $\mathfrak{H}$. An extension $\tilde{A}$ of A is called condensing if it is hermitian and $\mathfrak{D}(\tilde{A})\subset\overline{\mathfrak{D}(A)}$. If A has no condensing extension it is called semi-maximal. The article gives criteria for the existence of condensing extensions. For λ with $\mathfrak{J}\lambda\neq 0$, the following linear sets are defined: $\mathfrak{L}_\lambda$, the range of $(A-\lambda E)$; $\mathfrak{N}_\lambda=\mathfrak{H}\ominus\overline{\mathfrak{L}_\lambda}$, the λ-deficiency subspace of A; $\mathfrak{M}_\lambda$, the orthogonal projection into $\mathfrak{N}_\lambda$ of $\mathfrak{B}=\mathfrak{H}\ominus\mathfrak{D}$ and $\mathfrak{N}_\lambda'=\mathfrak{N}_\lambda\ominus\overline{\mathfrak{M}_\lambda}$, called the λ-semi-deficiency subspace of A. The dimension of $\mathfrak{N}_\lambda$ is called the λ semi-deficiency of A. The following isometric operators are defined: the usual Cayley transform $U_\lambda=(A-\bar{\lambda}E)(A-\lambda E)^{-1}$, and V_A, defined in $\mathfrak{M}_\lambda$ by $V_A P_{\mathfrak{N}_\lambda}h=P_{\mathfrak{N}_{\bar{\lambda}}}h$ for all h in $\mathfrak{B}$, where $P_{\mathfrak{N}}$ denotes the projector onto $\mathfrak{N}$.

The first main theorem, extending a well known theorem on extensions of hermitian operators [J. von Neumann, Math. Ann. **102**, 49–131 (1929)], states that any condensing extension of a closed hermitian operator A is defined for a domain of elements of the form $f=g+h-\bar{V}_A P_{\overline{\mathfrak{M}}_\lambda}h-UP_{\mathfrak{N}_\lambda'}h$ $(g\varepsilon\mathfrak{D}(A), h\varepsilon\mathfrak{A})$ by $\tilde{A}f=Ag+\bar{\lambda}h-\lambda(\bar{V}_A P_{\overline{\mathfrak{M}}_\lambda}h+UP_{\mathfrak{N}_\lambda'}h)$, where $\mathfrak{A}$ is a linear set such that $\mathfrak{A}\subset\mathfrak{N}_\lambda$, $\mathfrak{A}\cap\mathfrak{M}_\lambda=0$, and U an isometric operator with domain $P_{N_\lambda'}\mathfrak{A}$ and range in $\mathfrak{N}_{\bar{\lambda}}'$. The closed hermitian operator A is semi-maximal if and only if the isometric operator $U_\lambda\oplus V_A$ (the direct sum of the operators U_λ and V_A) has all $\mathfrak{H}$ either as domain or as range; equivalently, if and only if V_A is closed and one of the semi-deficiency indices of A (whose values as functions of λ are constant over each of the upper and lower half-planes) is zero; or, again, if and only if, for some nonreal λ, $\mathfrak{L}_\lambda+\mathfrak{B}=\mathfrak{H}$.

It is shown that a closed hermitian operator has a closed semimaximal condensing extension if and only if V_A is closed. An example demonstrates that, even if A and V_A are closed, A may have a closed condensing extension $\tilde{A}$ for which $V_{\tilde{A}}$ is not closed. If the space $\mathfrak{H}$ is imbedded in an extended space without altering the domain or values of A, the semi-deficiency indices and closure or nonclosure of V_A are unaltered, hence so are the possibilities of condensing extensions. Following M. A. Naĭmark [Izvestiya Akad. Nauk SSSR. Ser. Mat. **4**, 53–104 (1940); these Rev. **2**, 104] and extension $\tilde{A}$ of A defined in an extended space is called of second species if $\mathfrak{D}(A)=\mathfrak{H}\cap\mathfrak{D}(\tilde{A})$. The results of the present article are used to prove the theorem that every closed hermitian operator has a self-adjoint extension of second type, and other results proved by Naĭmark for operators with dense domain. *J. L. B. Cooper.*

Kadison, Richard V. Order properties of bounded self-adjoint operators. Proc. Amer. Math. Soc. **2**, 505–510 (1951).

Let $\mathfrak{A}'$ be a uniformly closed selfadjoint operator algebra on Hilbert space and $\mathfrak{A}$ the real linear space of selfadjoint elements of $\mathfrak{A}'$. The author studies $\mathfrak{A}$ as a partially ordered set (in the operator ordering) and the relationship of this ordering with the algebraic properties of $\mathfrak{A}'$, in particular to what extent $\mathfrak{A}'$ is noncommutative. For this purpose a partially ordered set is called an antilattice if the greatest lower bound of two elements exists only when they are comparable. And one of the main results can be stated as follows: A weakly closed selfadjoint operator algebra (containing I) is a factor if and only if the set of its selfadjoint elements is an antilattice. Among other results criteria for the existence of a greatest lower bound are obtained, especially in terms of the commutativity of the elements. As a corollary of one these results the following result of Sherman [see the following review] is obtained: A weakly closed selfadjoint operator algebra whose set of selfadjoint elements is a lattice, is commutative. *F. I. Mautner.*

Sherman, S. Order in operator algebras. Amer. J. Math. **73**, 227–232 (1951).

Soit S l'ensemble des éléments hermitiens d'une algèbre autoadjointe d'opérateurs d'un espace de Hilbert; on peut introduire dans S une relation d'ordre évidente; l'auteur démontre que si S, muni de cette relation d'ordre, est un

"lattis" (i.e. si, pour tout $H\epsilon S$, existe H^+ au sens de cette relation d'ordre) alors l'algèbre considérée est commutative.
R. Godement (Nancy).

Kaplansky, Irving. The structure of certain operator algebras. Trans. Amer. Math. Soc. **70**, 219–255 (1951).

L'auteur appelle *CCR*-algèbre toute algèbre autoadjointe uniformément fermée d'opérateurs d'un espace de Hilbert telle que, dans toute représentation unitaire irréductible $x \rightarrow U_x$ de cette algèbre, les U_x soient des opérateurs complètement continus; le but de l'auteur est évidemment de montrer que ces algèbres se comportent approximativement comme celles de dimension finie, ou comme les algèbres commutatives; dans cette catégorie rentrent les "algèbres de groupes" des groupes abéliens, ou compacts, et surtout—en vertu de résultats récents de Harish-Chandra—des groupes de Lie semi-simples connexes; c'est dire que cette classe d'algèbres est intéressante, et l'auteur en donne de nombreuses propriétés, dont voici les principales.

Tout d'abord, voici quelques généralizations du théorème d'approximation de Stone-Weierstrass. Soit X un espace compact, et supposons attaché à chaque $x\epsilon X$ une algèbre normée complète $\mathbf{A}_x$; soit $C(X)$ la famille des fonctions (complexes) continues sur X. Considérons une algèbre $\mathbf{A}$ de fonctions $f(x)$ à valeurs dans les diverses $\mathbf{A}_x$, et supposons vérifiées les conditions suivantes: (a) $\|f(x)\|$ est continue sur X; (b) $\mathbf{A}$ est complète pour la norme $\|f\| = \sup \|f(x)\|$; (c) pour tout x, les $f(x)$ $(f\epsilon\mathbf{A})$ forment tout $\mathbf{A}_x$; (d) $\mathbf{A}$ est un $C(X)$-module. L'auteur montre d'abord ceci: supposons que, pour tout $x\epsilon X$ et tout $z\epsilon\mathbf{A}_x$, z soit adhérent à $z\cdot\mathbf{A}_x$; alors, pour tout $f\epsilon\mathbf{A}$ et tout $\varphi\epsilon C(X)$, $\varphi\cdot f$ est adhérent à l'idéal $f\cdot\mathbf{A}$ de $\mathbf{A}$; de plus, dans cette hypothèse tout idéal à droite fermé I de $\mathbf{A}$ s'obtient en prenant dans chaque $\mathbf{A}_x$ un idéal à droite fermé I_x, I étant l'ensemble des f telles que $f(x)\epsilon I_x$ pour tout x (en particulier, si chaque $\mathbf{A}_x$ est simple pour les idéaux bilatères fermés, alors un idéal bilatère fermé de $\mathbf{A}$ est formé des f qui s'annulent sur un sous-ensemble fermé convenable de X). Supposons maintenant que chaque $\mathbf{A}_x$ soit isomorphe à l'algèbre des opérateurs complètement continus d'un espace de Hilbert, et soit $\mathbf{A}$ une *-algèbre de fonctions $f(x)$, $f(x)\epsilon\mathbf{A}_x$, avec les propriétés (a), (b) ci-dessus; supposons en outre que, pour $x\neq y$, il existe $f\epsilon\mathbf{A}$ pour laquelle $f(x)$ et $f(y)$ sont arbitrairement donnés; alors $\mathbf{A}$ est un $C(X)$-module, et tout idéal à droite fermé de $\mathbf{A}$ est intersection d'idéaux à droite maximaux.

Soit maintenant $\mathbf{A}$ une algèbre autoadjointe uniformément fermée; soit X l'ensemble des idéaux primitifs de $\mathbf{A}$, muni de la topologie de Jacobson: l'adhérence d'une partie E de X est l'ensemble des idéaux primitifs contenant $\bigcap_{P\epsilon E} P$. Pour chaque $P\epsilon X$, on a une algèbre $\mathbf{A}_P = \mathbf{A}/P$ (qui est aussi une algèbre autoadjointe uniformément fermée) et chaque $a\epsilon\mathbf{A}$ définit une fonction $a(P)$ dont la valeur en P est l'image de a dans $\mathbf{A}/P$. Il y a lieu de remarquer que X n'est pas localement compact, car l'axiome de Hausdorff n'est pas toujours vérifié; l'auteur montre d'abord que cet axiome équivaut à la propriété que, pour tout $a\epsilon\mathbf{A}$, la fonction $\|a(P)\|$ est continue sur X, et montre que c'est toujours le cas si, par exemple, les $\mathbf{A}/P$ sont toutes de dimension finie indépendante de P. Supposons maintenant que $\mathbf{A}$ soit une *CCR*-algèbre (i.e. que pour chaque P, $\mathbf{A}/P$ est l'algèbre de tous les opérateurs complètement continus d'un espace de Hilbert); alors X est un espace de Baire (toute intersection dénombrable d'ouverts partout denses est partout dense); de plus, on peut former dans $\mathbf{A}$ une série de composition I_ρ (i.e. une famille croissante d'idéaux bilatères fermés, commençant par 0 et terminant par $\mathbf{A}$, dépendant d'un ordinal ρ et telle que, pour tout nombre limite λ, I_λ soit engendré par les I_ρ précédents) telle que tous les quotients $I_{\rho+1}/I_\rho$ aient un "espace de structure" qui vérifie l'axiome de Hausdorff. Ce résultat implique l'analogue suivant du théorème de Weierstrass: soit $\mathbf{B}$ une sous-algèbre autoadjointe fermée d'une *CCR*-algèbre $\mathbf{A}$; supposons que, pour tout couple I', I'' d'idéaux à droite maximaux réguliers de $\mathbf{A}$ (avec $I'\neq I''$) $\mathbf{B}$ contienne un élément de I' non dans I''; alors $\mathbf{B}=\mathbf{A}$. L'auteur montre encore que, si $\mathbf{A}$ est une *CCR*-algèbre, alors il en est de même de $\mathbf{A}/I$ pour tout idéal autoadjoint fermé I, et de toute sous-algèbre autoadjointe fermée de $\mathbf{A}$. L'article se termine par quelques résultats relatifs aux algèbres algébriques.
R. Godement (Nancy).

Kaplansky, Irving. Projections in Banach algebras. Ann. of Math. (2) **53**, 235–249 (1951).

Le but de cet article est d'étudier une classe d'algèbres "abstraites" généralisant les anneaux d'opérateurs de Murray et von Neumann, et auxquelles les méthodes de ces auteurs sont susceptibles de s'appliquer. Non content de disposer déjà des C^*-algèbres, des W^*-algèbres, des B_p^*-algèbres, des *-algèbres et des *CCR*-algèbres, l'auteur introduit les AW^*-algèbres; ce sont les C^*-algèbres (=algèbres autoadjointes uniformément fermées d'opérateurs d'un espace de Hilbert) vérifiant les deux axiomes suivants: (A) toute famille de projecteurs deux à deux orthogonaux appartenant à l'algèbre donnée $\mathbf{A}$ possède une borne supérieure dans $\mathbf{A}$; (B) toute sous-algèbre autoadjointe commutative maximale de $\mathbf{A}$ est engendrée par ses projecteurs. L'auteur montre tout d'abord que, si $\mathbf{A}$ est une AW^*-algèbre, il en est de même de son centre, de ses sous-algèbres autoadjointes commutatives maximales, et de toute sous-algèbre de la forme $e\mathbf{A}e$, où e est un projecteur de $\mathbf{A}$. Soient maintenant e, f deux projections dans $\mathbf{A}$; on écrit $e\sim f$ s'il existe $x\epsilon\mathbf{A}$ tel que $e=xx^*$, $f=x^*x$; on écrit $e \dashv f$ si $e\sim g$ avec $g\leqq f$; on dit que e est finie si $f\sim e$, $f\leqq e$ implique $f=e$. Tout le reste de l'article consiste à étendre (par des méthodes souvent ingénieuses) aux AW^*-algèbres un grand nombre des propriétés démontrées dans les cas "concrets" par Murray et von Neumann, avec naturellement les modifications requises par le fait qu'on ne suppose pas le centre de $\mathbf{A}$ réduit aux scalaires. En ce qui concerne l'utilité d'une telle étude, elle n'apparait pas encore très clairement, sinon pour simplifier éventuellement les démonstrations de Murray et von Neumann; il est en tous cas permis de croire qu'une étude approfondie des anneaux d'opérateurs (au sens classique), telle que Dixmier par exemple la poursuit, risque d'être plus immédiatement utilisable (il faut noter à ce propos l'absence de toute référence aux travaux de Dixmier, lesquels comprennent les notions générales d'équivalence introduites par l'auteur, la décomposition globale d'un anneau en parties de types I, II et III, et beaucoup d'autres résultats considérablement plus difficiles à établir, en particulier une théorie à peu près complète des traces sur les anneaux arbitraires; ces résultats, contemporains de ceux de l'auteur, doivent paraitre incessamment).
R. Godement (Nancy).

Yood, Bertram. Banach algebras of continuous functions. Amer. J. Math. **73**, 30–42 (1951).

Soient $\mathbf{A}$ une algèbre normée complète commutative avec unité, et X un espace topologique; l'auteur étudie les idéaux maximaux de l'algèbre normée $C(X, \mathbf{A})$ formée des fonctions continues bornées sur X à valeurs dans $\mathbf{A}$. Soit $\mathfrak{M}$ l'espace

(compact) des idéaux maximaux de **A**, et supposons tout d'abord que $\mathbf{A}=C(\mathfrak{M})$; alors, si X est complètement régulier, l'espace des idéaux maximaux de $C(X, \mathbf{A})$ est homéomorphe à la compactification (au sens de Čech) de $X\times\mathfrak{M}$, pour la raison assez évidente que $C(X, \mathbf{A})$ s'identifie à $C(X\times\mathfrak{M})$. Dans le cas où X est en outre normal ainsi que $X\times\mathfrak{M}$, l'auteur donne des résultats plus précis et aussi plus compliqués sur ces idéaux maximaux. Enfin, dans le cas où **A** est arbitraire et X compact, l'auteur démontre que l'espace des idéaux maximaux de $C(X, \mathbf{A})$ est $X\times\mathfrak{M}$ en faisant l'hypothèse que, pour tout idéal maximal I de $C(X, \mathbf{A})$, il existe un $t\epsilon X$ et un idéal maximal M de **A** tels que $\mathbf{f}\epsilon I$ implique $\mathbf{f}(t)\epsilon M$; mais il est évident que cette hypothèse est toujours réalisée (en effet, un idéal maximal I de $C(X, \mathbf{A})$ est le noyau d'un homomorphisme $\mathbf{f}\rightarrow\chi(\mathbf{f})$ de $C(X, \mathbf{A})$ dans le corps complexe; si $\mathbf{f}(x)=\mathbf{a}$, où $\mathbf{a}\epsilon\mathbf{A}$ est fixe, on a nécessairement $\chi(\mathbf{f})=\mathbf{a}(M)$ où M est un certain idéal maximal de **A**; si au contraire $\mathbf{f}(x)=f(x)\mathbf{e}$, où $f(x)$ est continue à valeurs complexes, on a évidemment $\chi(\mathbf{f})=f(t)$, où t est un certain point de X; comme les fonctions de la forme $\sum f_i(x)\mathbf{a}_i$ sont partout denses dans $C(X, \mathbf{A})$, il s'ensuit immédiatement qu'on a $\chi(\mathbf{f})=\mathbf{f}(t)(M)$ pour toute $\mathbf{f}\epsilon C(X, \mathbf{A})$, ce qui démontre l'hypothèse en question). *R. Godement* (Nancy).

Halperin, Israel, and Nakano, Hidegoro. Discrete semi-ordered linear spaces. Canadian J. Math. 3, 293–298 (1951).

Let R be a semi-ordered linear space. An element b of R is said to be discrete if for each x such that $|x|\leqq|b|$ there exists a real number β for which $x=\beta b$. The space R itself is said to be discrete if it is universally continuous (i.e. conditionally complete) and if it has a complete set of discrete elements. The conjugate space $\bar{R}$ to a universally complete R consists of all linear functionals L such that $\inf_{\lambda\epsilon\Lambda}|L(a_\lambda)|=0$ if $\bigcap a_\lambda=0$ and if given λ_1, λ_2 there exists a λ_3 such that $a_{\lambda_3}\leqq a_{\lambda_1}\cap a_{\lambda_2}$. Then R is said to be semi-regular if $\bar{R}$ is total. The notion of w-convergence is defined by $\lim_{n\to\infty}\bar{x}(a_n)=\bar{x}(a)$ for all $\bar{x}\epsilon\bar{R}$ whereas $|w|$-convergence means $\lim_{n\to\infty}\bar{x}(|a_n-a|)=0$ for all $\bar{x}\epsilon\bar{R}$. The purpose of the paper is to prove that each of the following is necessary and sufficient in order that R should be discrete. (a) R is semi-regular and w-convergence coincides with $|w|$-convergence. (b) R is semi-regular and star individual convergence coincides with individual convergence. (c) R is semi-regular and $|w|$-convergence implies individual convergence.

R. Phillips (Los Angeles, Calif.).

Sasaki, Usa. On an axiom of continuous geometry. J. Sci. Hiroshima Univ. Ser. A. 14, 100–101 (1950).

Von Neumann's continuity axiom for a lattice L can be formulated as the following (α) and its dual (α^*). (α): If $a_\alpha\uparrow a$ then $a_\alpha\cap b\uparrow a\cap b$ for any b where the α are any ordered set of indices. The writer points out that in a complete lattice (α) implies the same axiom but with the α any directed set of indices. The continuity axiom thus implies that meet and join of two elements are continuous in both elements simultaneously with respect to convergence not merely of the usual sequential-order kind but even of the directed-sequential-order kind. *I. Halperin.*

***Fantappiè, Luigi. La teoria dei funzionali analitici, le sue applicazioni e i suoi possibili indirizzi.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 223–279, Rome, 1943.

Der Autor gibt einen zusammenfassenden Bericht über die Entwicklung und den aktuellen Stand der Theorie der analytischen Funktionale. Im ersten Kapitel werden die verschiedenen Funktionalbegriffe und ihre gegenseitigen Beziehungen historisch beleuchtet. Im zweiten Kapitel wird die Theorie der analytischen Funktionale entwickelt. Entgegen einer ersten Fassung werden als Argumente lokal-analytische Funktionen und nicht analytische Funktionen im Sinne von Weierstrass genommen. Dadurch wird einerseits die Vieldeutigkeit der Argument-funktionen vermieden und andererseits eine einfachere Struktur des Funktionalraumes erhalten, dessen lineare Gebiete eineindeutig den abgeschlossenen Punktmengen des Raumes von n komplexen Variablen oder der ihm entsprechenden Segre-Mannigfaltigkeit V_{2n} zugeordnet sind. Allerdings ist die eingeführte Topologie sehr schwach, da eine Funktion und jede ihrer lokal-analytischen Fortsetzungen als von einander verschiedene Punkte aufgefasst werden müssen. Nach den verschiedenen Indikatrixdarstellungen eines linearen Funktionals wird ein allgemeines Theorem über die Singularitäten einer transformierten Funktion gegeben, das die Sätze von Hadamard und Hurwitz als Spezialfälle enthält. Anschliessend werden die Ableitungen und die Reihenentwicklung eines nichtlinearen Funktionals erklärt. Im dritten Kapitel wird die Theorie zum Aufbau eines exakten Kalküls der linearen Operatoren verwendet. Als Beispiele werden Matrizenfunktionen und Funktionen des unbestimmten Integrals behandelt. Im vierten Kapitel werden vier Methoden zum Lösen von partiellen Differentialgleichungen entwickelt. Die Arbeit schliesst mit einem Kapitel über weitere Anwendungsmöglichkeiten und Forschungsrichtungen.

H. G. Haefeli (Cambridge, Mass.).

Fantappiè, Luigi. L'analisi funzionale nel campo complesso e i nuovi metodi d'integrazione delle equazioni a derivate parziali. Rivista Mat. Univ. Parma 1, 117–120 (1950).

Expository article with an extensive bibliography.

H. G. Haefeli (Cambridge, Mass.).

Carafa, Mario. L'indicatrice dei funzionali analitici polinomiali. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 8, 200–210 (1949).

Der Autor beweist, dass der Wert eines analytischen Polynomial-Funktional in jedem Punkt einer linearen Umgebung seines Definitionsgebietes mit Hilfe zweier Integrationen gefunden werden kann, sobald man seinen Wert längs Stückes einer gewissen analytischen Linie in der linearen Umgebung kennt. Diese Werte bilden definitionsgemäss eine analytische Funktion des Kurvenparameters, welche lokale Indicatrix genannt wird. Abschliessend wird gezeigt, dass sich die gefundene Darstellung für lineare Funktionale auf eine einzige Integration reduziert und mit der Fundamentalformel der letzteren übereinstimmt.

H. G. Haefeli (Cambridge, Mass.).

Dunford, Nelson. An individual ergodic theorem for non-commutative transformations. Acta Sci. Math. Szeged 14, 1–4 (1951).

The ergodic theorems were originally concerned with a one-parameter family of measure-preserving transformations of a measure space. The extension to an n-parameter abelian family of measure-preserving transformations was given by N. Wiener [Duke Math. J. 5, 1–18 (1939)] with the view of application to the homogeneous chaos. The present paper deals with the n-parameter non-abelian case which, the author says, was proposed by H. E. Robbins. The result reads as follows. Let $\varphi_1, \cdots, \varphi_k$ be one-to-one measure-preserving maps of a measure space Ω of finite measure

onto itself, and let $p>1$. Then for every $f\epsilon L_p(\Omega)$ the multiple sequence

$$(m_1\cdots m_k)^{-1}\sum_{\nu_1=0}^{m_1-1}\cdots\sum_{\nu_k=0}^{m_k-1} f(\varphi_1^{\nu_1}\cdots\varphi_k^{\nu_k}\omega)$$

is convergent, as $m_1, \cdots, m_k\to\infty$ independently, almost everywhere as well as in the mean of order p, and is dominated by a function in $L_p(\Omega)$. The theorem holds good also for $f\epsilon L_1(\Omega)$ if $\int_\Omega|f(\omega)|\log^+|f(\omega)|d\omega<\infty$ and if $k=2$. The proof is carried out by induction with respect to k, combining the mean ergodic theorem in Banach spaces [F. Riesz, J. London Math. Soc. **13**, 274–278 (1938); K. Yosida and S. Kakutani, Proc. Imp. Acad. Tokyo **14**, 292–294, 295–300 (1938)] with the principles of the deduction of the individual ergodic theorems from the mean ergodic theorems [N. Wiener, loc. cit.; K. Yosida: Jap. J. Math. **17**, 31–36 (1940); N. Dunford and D. S. Miller, Trans. Amer. Math. Soc. **60**, 538–549 (1946); these Rev. **2**, 105; **8**, 280]. At the end of the paper it is noted that A. Zygmund also obtained the same result for the case $p=1$. *K. Yosida.*

Calculus of Variations

★Lavrent'ev, M. A., i Lyusternik, L. A. Kurs variacionnogo isčisleniya. [Course in the Calculus of Variations]. 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 296 pp.

I. Elementary methods of solution of extremal problems. II. The method of variations. III. Generalization of the simplest problems. IV. Admissible curves with variable end points. Discontinuous problems. V. Conditional extremum. VI. Variational problems in parametric form. VII. The theory of fields. VIII. Sufficient conditions for strong and weak extrema. IX. Linear variational problems.

Table of contents.

Picone, Mauro. Due conferenze sui fondamenti del "Calcolo delle variazioni." Giorn. Mat. Battaglini (4) **4(80)**, 50–79 (1951).

For a real function f defined on a space X, the following three problems are considered. Problem A: to determine the greatest lower bound $e(f, X)$ of f on X. Problem B: to determine whether a minimizing element for f exists in X. Problem C: to determine the set of all minimizing elements. When the set X is the union of a nondecreasing sequence of compact sets X_n, the solution of problems A and B is reduced to the solution of these problems for the sets X_n, and similarly for problem C. Obviously, $e(f, X)=\lim e(f, X_n)$. If f is lower semicontinuous on each X_n, f has a minimum on X if and only if $e(f, X)=e(f, X_n)$ for $n\geqq\mu$. In that case, the set of minimizing elements for f on X is the union of the corresponding sets for X_n where $n\geqq\mu$. The major part of the paper is taken up with the discussion of examples. The first of these is the general calculus of variations problem for multiple integrals involving partial derivatives up to those of order ν. The derivatives of νth order are supposed to satisfy a Hölder condition of a type called "perimetric". Under suitable conditions the corresponding function space is shown to be the union of a sequence of compact sets, and the integral is obviously continuous in terms of the metric adopted. The parametric curve problem of the calculus of variations and some other examples are also discussed. In his discussion of the topological basis, the author's terminology and notations unfortunately diverge from standard practice in a number of instances. *L. M. Graves.*

Popoff, Kyrille. Sur une propriété des extrémales et le théorème de Jacobi. C. R. Acad. Sci. Paris **230**, 1032–1033 (1950).

The present paper is concerned with an interpretation of conjugate points. After a transformation of variables, a suitable one parameter family of extremals is expressed in the form:

$$\eta=\eta_0(\xi)+\eta_1(\xi)\tan\tfrac{1}{2}\alpha+\cdots.$$

The conjugate points are determined by the properties of $\eta_1(\xi)$. The author proposes to construct conjugate points by a method based only on the properties of $\eta_0(\xi)$.

M. R. Hestenes (Los Angeles, Calif.).

Davies, E. T. On the second variation of the volume integral when the boundary is variable. Quart. J. Math., Oxford Ser. (2) **1**, 248–252 (1950).

L'auteur donne une extension au cas où la frontière est variable de sa formule exprimant la variation seconde d'une intégrale multiple [Quart. J. Math., Oxford Ser. (1) **13**, 58–64 (1942); ces Rev. **4**, 115]. Cette formule qui est une transformation d'une formule initialement donnée par de Donder [Théorie invariantive du calcul des variations, nouvelle ed., Gauthier-Villars, Paris, 1935, formule (172)] présente le gros avantage de mettre directement en évidence les éléments géométriques du problème. Elle entraine comme corollaire, dans le cas d'une intégrale simple, une interprétation donnée par Morse [The Calculus of Variations in the Large, Amer. Math. Soc. Colloquium Publ., v. 18, New York, 1934, formule (3–19)] de la seconde forme fondamentale d'une hypersurface. *A. Lichnerowicz* (Paris).

Nickel, Karl. Lösung eines Minimumproblems der Tragflügeltheorie. Z. Angew. Math. Mech. **31**, 72–77 (1951). (German. English, French, and Russian summaries)

Let A_n, $n=1, \cdots, N$, be fixed real numbers and $h_n(x)$ fixed real functions (say, piecewise continuous). The problem is to find $\Gamma(x)$, $|x|\leqq 1$, $\Gamma(-1)=\Gamma(1)=0$, such that $\int_{-1}^1\Gamma(x)w(x)dx$ is a minimum when the conditions

$$\int_{-1}^1\Gamma(x)h_n(x)dx=A_n, \qquad n=1, \cdots, N,$$

are satisfied, and where $w(x)=(4\pi)^{-1}\int_{-1}^1\Gamma'(y)(x-y)^{-1}dy$ (defining the integral by the Cauchy principal value). By a change of variable this is reduced to finding a function $f(s)$ minimizing $\int_0^\pi\int_0^\pi f(s)f(t)S(s, t)dsdt$ when

$$\int_0^\pi\int_0^\pi f(s)h_n(t)S(s, t)dsdt=A_n,$$

where $S(s, t)=2\pi^{-1}\sum n^{-1}\sin ns\sin nt$. The author shows that there exists a minimum with

$$\Gamma(x)=\sum a_n\int_{-1}^1 h_n(y)\log\left|\frac{1-xy+[(1-x)^2(1-y)^2]^{\frac{1}{2}}}{x-y}\right|dy$$

if the h_n are linearly independent. A short table of this integral is given for several choices of h. [See also an earlier paper, Math. Z. **53**, 21–52 (1950); these Rev. **12**, 423.]

J. V. Wehausen (Providence, R. I.).

Theory of Probability

Menger, Karl. Probabilistic theories of relations. Proc. Nat. Acad. Sci. U. S. A. **37**, 178–180 (1951).

If we interpret "equal" as "indistinguishable" then the relation of equality need not be transitive. The author proposes a quantitative definition $E(a, b)$, the probability that a and b be equal. Postulates: $E(a, a) = 1$; $E(a, b) = E(b, a)$; $E(a, b)E(b, c) \leqq E(a, c)$. Corresponding to each a let A be the set of all x such that $E(a, x) = 1$. Define $E(A, B) = E(a, b)$ for any a in A, b in B. Putting $-\log E(A, B) = d(A, B)$ it is easily seen that this defines a metric in the space of A, and conversely. *K. L. Chung* (Ithaca, N. Y.).

Menger, Karl. Probabilistic geometry. Proc. Nat. Acad. Sci. U. S. A. **37**, 226–229 (1951).

To every ordered pair (a, b) of a set S we associate a distribution function $\Delta_{ab}(x)$ = the probability that the distance from a to b be $<x$. Postulates: $\Delta_{aa}(x) = 1$ for every $x > 0$; $\Delta_{ab}(x) = 0$ for every $x \leqq 0$; $\Delta_{ab}(x) = \Delta_{ba}(x)$; $[\Delta_{ab} * \Delta_{bc}](x) \leqq \Delta_{ac}(x)$, for every x where $*$ denotes convolution. This defines a probabilistic metric which includes the ordinary metric as a special case. Various types of "distinguishability" of a and b are defined according to the behavior of $\Delta_{ab}(x)$ as x decreases to 0. [On p. 227, line 7 read "least upper bound" for "greatest lower bound".] Concrete examples and possible applications (to psychophysics etc.) are discussed. *K. L. Chung* (Ithaca, N. Y.).

Menger, Karl. Ensembles flous et fonctions aléatoires. C. R. Acad. Sci. Paris **232**, 2001–2003 (1951).

The author continues his remarks in the two preceding notes. Example: To every x in $\langle a, b\rangle$ assign a function $f(x, t)$ = the probability that $x < t$. For each t, $y = f(x, t)$ is a curve C_t. The author proposes to find the area between the x-axis and the set of curves C_t. In customary lingo this should be the integral $\int_a^b x(s)ds$ where the $x(s)$, $a \leqq s \leqq b$, are independent (this assumption is essential but not stated), normally distributed (this assumption is unnecessary) random variables with mean $\mu(s)$ and variance $\sigma^2(s)$. It is shown that the Riemann-Darboux sums for this integral converge in probability (in mean, therefore) to $\int_a^b \mu(s)ds$ if both this and $\int_a^b \sigma^2(s)ds$ are finite. This is an easy variant of the weak law of large numbers. *K. L. Chung.*

Schützenberger, Marcel-Paul. Une généralisation de la notion de valuation pour les treillis quelconques et son application aux distributions de la statistique quantique. C. R. Acad. Sci. Paris **232**, 1805–1807 (1951).

A function defined on a lattice T, with values in a commutative ring, is called a generalized valuation if it satisfies the usual valuation functional equation in the elements a, b of the lattice whenever $[a \cap b, a \cup b]$ is isomorphic to $[a \cap b, a] \times [a \cap b, b]$. If T is the lattice of partitions of a space, the generalized valuations lead to a function with the properties of information relative to the differentiation between sets of the partition, and conversely. See the author's paper [same C. R. **232**, 925–927 (1951); these Rev. **12**, 623] for a discussion of information functions. The logarithm of the probability of a specified partition of a system of particles into cells defines such a partition function, and this fact leads to a formula for the most general probability distribution of particles into cells. *J. L. Doob* (Urbana, Ill.).

Van Dantzig, D. Une nouvelle généralisation de l'inégalité de Bienaymé. Ann. Inst. H. Poincaré **12**, 31–43 (1951).

The author establishes an inequality which generalizes those of Meidell and Camp. The method of proof is based upon an idea propounded by von Mises and de Jongh: replace $P(t) = P[|X| \geqq t]$ by a convenient $Q(t) = P[|Y| \geqq t]$ such that $P(t) \geqq Q(t)$ for $t > 0$ and $P(t_0) = Q(t_0)$ for a value $t_0 > 0$ of t. *M. Loève* (Berkeley, Calif.).

Sapogov, N. A. The stability problem for a theorem of Cramér. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 205–218 (1951). (Russian)

In a previous paper [Doklady Akad. Nauk SSSR (N.S.) **73**, 461–462 (1950); these Rev. **12**, 191] the author stated the theorem that if the sum of two independent random variables is approximately Gaussian then each summand is also approximately Gaussian. In the present paper this result is proved, and the inequalities given already, which make the above statement precise, are improved slightly. *J. L. Doob* (Urbana, Ill.).

Rényi, Alfréd. On some problems concerning Poisson processes. Publ. Math. Debrecen **2**, 66–73 (1951).

The Poisson process is derived from postulates of the usual type. It is shown that if every event in a Poisson process starting at time 0 is the birth of an individual whose life is a random variable (with distribution which may depend on the moment of birth), then the number of individuals alive at time t has a Poisson distribution. *J. L. Doob* (Urbana, Ill.).

Kendall, David G. On non-dissipative Markoff chains with an enumerable infinity of states. Proc. Cambridge Philos. Soc. **47**, 633–634 (1951).

Let p_{ij} denote the transition probability that a simple Markoff system of enumerable infinite number of possible states will move from ith state into jth state. As is known [A. Kolmogoroff, Mat. Sbornik N.S. **1**(**43**), 607–615 (1936); K. Yosida and S. Kakutani, Jap. J. Math. **16**, 47–55 (1939); these Rev. **1**, 62], the time average

$$\lim_{n\to\infty} n^{-1}\sum_{k=1}^{\infty} p_{ij}^{(k)} = \pi_{ij} \quad \left(p_{ij}^{(k)} = \sum_{s=1}^{\infty} p_{is}^{(k-1)} p_{sj},\ p_{ij}^{(1)} = p_{ij}\right)$$

exists. As a supplement to a paper by E. G. Foster [same Proc. **47**, 77–85 (1951); these Rev. **12**, 620], the author proves the result: $\sum_{j=1}^{\infty} \pi_{ij} = 1$ for every i if there exists $(w_1, w_2, \cdots)$ such that $\limsup_{i\to\infty} w_i = \infty$, $w_i \geqq \sum_{j=1}^{\infty} p_{ij} w_j$ $(i = 1, 2, \cdots)$. *K. Yosida* (Nagoya).

Kendall, David G. Random fluctuations in the age-distribution of a population whose development is controlled by the simple "birth-and-death" process. J. Roy. Statist. Soc. Ser. B. **12**, 278–285 (1950).

The characteristic functional of the birth and death process, assuming constant birth and death rates, is evaluated. This result has been extended by Bartlett and Kendall [Proc. Cambridge Philos. Soc. **47**, 65–76 (1951); these Rev. **12**, 620] to allow immigration. *J. L. Doob.*

Mathematical Statistics

Pearson, E. S., and Merrington, Maxine. Tables of the 5% and 0.5% points of Pearson curves (with argument β_1 and β_2) expressed in standard measure. Biometrika **38**, 4–10 (1951).

Bose, P. K. Corrigenda: On the construction of incomplete probability integral tables of the classical D^2-statistic. Sankhyā 11, 96 (1951).

See Sankhyā 8, 235–248 (1947); these Rev. 9, 620.

Pearson, E. S., and Hartley, H. O. Charts of the power function for analysis of variance tests, derived from the non-central F-distribution. Biometrika 38, 112–130 (1951).

The authors construct charts indicating values of $P(F'\geqq F_\alpha)$ where F' is the noncentral F statistic and F_α is defined by $P(F\geqq F_\alpha)=\alpha$. For each α, the above probability depends on ν_1, ν_2 and φ where ν_1 and ν_2 are the degrees of freedom associated with the numerator and denominator of F, respectively, and φ is Tang's noncentrality parameter [Statist. Res. Mem. London 2, 126–149 (1938)]. Charts are constructed for $\alpha=.05$ and $.01$, $\nu_1=1(1)8$ and $\nu_2=6(1)10$, 12, 15, 20, 30, 60, and ∞. These charts are largely based on Tang's tables. *H. Chernoff.*

Krishnamoorthy, A. S. On the orthogonal polynomials associated with Student's distribution. Sankhyā 11, 37–44 (1951).

The author derives explicit formulas for the finite set of orthogonal polynomials associated with Student's distribution, $B_\nu(1+t^2/\nu)^{-(\nu+1)/2}$ ($\nu=$integer>0, $-\infty<t<+\infty$). With $0\leqq 2m<\nu$ they are

$$\varphi_m(t)=b_m\sum_{r=0}^{\frac{1}{2}m}(-1)^r\frac{2^{2r}}{\nu^r(2r)!}\times\left\{\Gamma(\tfrac{1}{2}m+1-r)\Gamma\left(\frac{\nu-m}{2}+1-r\right)\right\}^{-1}t^{2r}\quad(m\text{ even}),$$

$$\varphi_m(t)=b_m\sum_{r=0}^{\frac{1}{2}(m-1)}(-1)^r\frac{2^{2r}}{\nu^r(2r+1)!}\times\left\{\Gamma(\tfrac{1}{2}(m-1)+1-r)\Gamma\left(\frac{\nu-m-1}{2}+1-r\right)\right\}^{-1}t^{2r+1}\quad(m\text{ odd}).$$

These polynomials are then used to generalize Student's distribution in exactly the same way that Hermite polynomials are used in the Gram-Charlier generalization of the normal distribution. The paper continues the work of V. Romanovsky [Biometrika 16, 106–116 (1924); C. R. Acad. Sci. Paris 188, 1023–1025 (1929)]. A factor ν^{-m} is lost from the last members of equations (2.9) and (2.10). *S. W. Nash* (Vancouver, B. C.).

Chapman, Douglas G. Some properties of the hypergeometric distribution with applications to zoölogical sample censuses. Univ. California Publ. Statist. 1, 131–159 (1951).

The properties referred to are derived for estimation and testing hypotheses regarding parameters associated with a generalized hypergeometric distribution. From each of r finite populations, assumed to consist of subpopulations, a sample of fixed size is drawn without replacement and the numbers of sample elements associated with the respective subpopulations are noted. Each ratio, p_{ij}, of the size of the jth subpopulation to the size of the ith population is assumed to be a known function, satisfying certain regularity conditions, of unknown parameters

$$\theta_1,\cdots,\theta_m\quad(i=1,\cdots,r;\ j=1,\cdots,v_i).$$

BAN estimates and tests associated with them are obtained by adaptation of methods recently developed by Neyman [Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, pp. 239–273, Univ. of California Press, Berkeley-Los Angeles, 1949; these Rev. 10, 388]. Some results derived in this adaptation are of greater generality than required in regard to the hypergeometric distribution. The bias and variability of several estimates of population size (also subpopulation size) from a single sample census are treated in detail; tables relevant to these quantities are given. Various tests for making certain comparisons of population sizes are presented together with tables giving information on the power of some of the tests. *D. F. Votaw, Jr.* (New Haven, Conn.).

Sato, Ryoichiro. "r" distributions and "r" tests. Ann. Inst. Statist. Math., Tokyo 2, 91–124 (1951).

The distribution whose density function is

$$p(r)=[\Gamma(\tfrac{1}{2}(f+1))/\pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}f)](1-r^2)^{\frac{1}{2}f-1}$$

is called an "r" distribution with f degrees of freedom. The author uses geometrical methods to prove a number of theorems of which the following is a typical example. Theorem. If $(x_1,x_2,\cdots,x_N)$ is a random sample from a normal population and if $(a_1,a_2,\cdots,a_N)$ is any set of real numbers such that the a's are not all equal then $\sum_{i=1}^N(x_i-x)(a_i-a)/[\sum_{i=1}^N(x_i-x)^2\sum_{i=1}^N(a_i-a)^2]^{\frac{1}{2}}$ has an r-distribution with $N-2$ degrees of freedom. By specializing the a's a number of tests are easily obtained for many of the usual hypotheses such as the comparison of two sample means etc. The author points out the equivalence of these tests to the more familiar ones based on statistics which follow Student's t-distribution. The later theorems and example deal with standard normal regression problems. *D. G. Chapman* (Seattle, Wash.).

Mood, A. M. On the distribution of the characteristic roots of normal second-moment matrices. Ann. Math. Statistics 22, 266–273 (1951).

This paper gives a somewhat more elementary derivation of the distributions of characteristic roots of normal second-moment matrices than is generally found in the literature. The author derives the normalizing constants associated with the distributions in an interesting and rigorous manner. This work is closely connected with that of Hotelling on canonical correlations for two sets of variates [Biometrika 28, 321–377 (1936)]. *B. Epstein* (Detroit, Mich.).

David, S. T., Kendall, M. G., and Stuart, A. Some questions of distribution in the theory of rank correlation. Biometrika 38, 131–140 (1951).

The exact distribution of Spearman's rank correlation coefficient r_s in the null case is tabulated for sample sizes $n=9$ and 10. [For $n<9$ see M. G. Kendall, S. F. H. Kendall, and B. B. Smith, Biometrika 30, 251–273 (1939).] The moments and cumulants of r_s in the null case up to order 8 are given. The distribution functions of r_s and of Kendall's coefficient t are approximated by the first terms of Edgeworth's series. The product moment correlation between r_s and t in samples from a bivariate normal population is examined. *W. Hoeffding* (Chapel Hill, N. C.).

Gayen, A. K. The frequency distribution of the product-moment correlation coefficient in random samples of any size drawn from non-normal universes. Biometrika 38, 219–247 (1951).

Let $F(x, y)$ be specified by a bivariate Edgeworth surface

$$F(x,y)=\left\{1+\sum_{i+j=3,4,6}\sum A_{ij}\frac{(-1)^{i+j}}{i!j!}\left(\frac{\partial}{\partial x}\right)^i\left(\frac{\partial}{\partial y}\right)^j\right\}\varphi(x,y),$$

where $\varphi(x, y)$ represents the standardized normal bivariate distribution. Let ρ and r represent respectively the population value and the sample value of the coefficient of correlation. The author obtains explicitly the distribution of r and its moment characteristics in samples of N from $F(x, y)$. In case $\rho\neq 0$, the normal-theory law is affected by changes in the population form no matter how large N may be. But if $\rho=0$ the effect of non-normality is not serious. The transformed variable $z=\frac{1}{2}\{\log(1+r)-\log(1-r)\}$ is considerably influenced by the non-normality of the population if $\rho\neq 0$, but the disturbance is largely in the mean and variance of z. For reasonable values of N the assumption of the normality of z is a remarkably good approximation. Many excellent numerical tables, graphs, and examples illustrate the theory, particularly for $N=11$, $\rho=0$, .8; and for $N=21$, $\rho=.8$. In case $F(x, y)=\varphi(x, y)$, i.e. for x and y distributed normally, the author corrects the formulas for the moments of z as given by R. A. Fisher [Metron 1, no. 4, 3–32 (1921)].

L. A. Aroian (Culver City, Calif.).

Fraser, D. A. S. Normal samples with linear constraints and given variances. Canadian J. Math. 3, 363–366 (1951).

The author considers the problem of the existence and construction of a set of normal variates connected by given linear relations and having given variances.

W. Hoeffding (Chapel Hill, N. C.).

Hemelrijk, J. Statistical determination of the linear relation between two physical quantities. Nederl. Tijdschr. Natuurkunde 17, 147–158 (1951). (Dutch. English summary)

To estimate and give confidence regions for an exact relation between two variables subject to error, a ranking method is employed, which assumes independence but not normality.

H. Wold (Uppsala).

Sichel, Herbert S. The estimation of the parameters of a negative binomial distribution with special reference to psychological data. Psychometrika 16, 107–127 (1951).

The author deals with the efficiency of the method of moments in estimating the parameters of the negative binomial distribution and with a computational method of obtaining the maximum likelihood estimates. Appropriate tables and charts are included. Applications are given to investigate absence proneness and accident proneness etc.

H. Chernoff (Stanford University, Calif.).

Dalenius, Tore. The problem of optimum stratification. Skand. Aktuarietidskr. 33, 203–213 (1950).

The variate y follows the frequency function $f(y)$, $a<y<b$. Points of division $a, y_1, y_2, \cdots, y_{k-1}, b$, with $y_{i-1}<y_i$, are to be set up to form k strata for the estimation of the mean value of y by a stratified random sample. The estimate is $\sum p_i\bar{y}_i$, where p_i is the area of the ith stratum and $\bar{y}_i$ is the mean of the sample from the ith stratum. If the sampling ratio is constant in all strata, the points of division which give the estimate of minimum error variance are shown to satisfy the relations $2y_i=\mu_i+\mu_{i+1}$, where μ_i is the mean of the stratum bounded by y_{i-1} and y_i. If the sampling ratios are optimum in the sense of Neyman, the best points of division satisfy the relations

$$\frac{\sigma_i^2+(y_i-\mu_i)^2}{\sigma_i}=\frac{\sigma^2_{i+1}+(y_i-\mu_{i+1})^2}{\sigma_{i+1}}$$

where σ_i^2 denotes the within-stratum variance.

W. G. Cochran (Baltimore, Md.).

David, F. N., and Johnson, N. L. The effect of non-normality on the power function of the F-test in the analysis of variance. Biometrika 38, 43–57 (1951).

Given a one-way classification, the authors compute the moments of the statistic

$$\sum_{t=1}^{s}n_t(\bar{z}_{t.}-\bar{z}_{..})^2-\frac{s-1}{N-s}F_\alpha\sum_{t=1}^{s}\sum_{i=1}^{n_t}(z_{ti}-\bar{z}_{t.})^2$$

in terms of given cumulants of the distributions of the z_{ti}, all variables corresponding to identical values of t having the same cumulants. By fitting various frequency distributions to these moments, the approximate power of the F-test with respect to alternatives characterized by the given cumulants may be found.

G. E. Noether.

Barankin, E. W., and Gurland, J. On asymptotically normal, efficient estimators. I. Univ. California Publ. Statist. 1, 89–129 (1951).

Neyman pointed out [Proc. Berkeley Symposium on Math. Statistics and Probability, U. of California Press, 1949, pp. 239–273; these Rev. 10, 388] that, since the consistency, asymptotic normality, and minimal asymptotic variance (efficiency) of maximum likelihood estimators (under certain regularity conditions) constitute "the only rational basis for preferring them to other estimates," any other estimator possessing these properties will, "as far as asymptotic properties are concerned, be just as good an estimate." Certain estimators of the latter type, called BAN estimators, have the advantage that they will often be easier to calculate, requiring only linear methods in many cases. The present authors generalize Neyman's work on the multinomial problem to a broader class Π_1 of families P of distributions characterized by a finite number of real parameters. The conditions and definitions of the paper are too lengthy to detail, but the essential results are as follows: A method M_0 is given for constructing (for each sample size) an estimator of the parameters of the given family P. In general many such estimators may be constructed using M_0. A class of consistent and asymptotically normal estimators generated by M_0 is characterized. If P satisfies a Koopman form with certain regularity conditions, a subclass of the above class contains estimators which are efficient for all values of the parameters. A second method of generating such estimators is given, which under certain restrictions requires only the solution of linear equations. Examples are given which demonstrate the simplicity of this method over that of maximum likelihood. Fundamental to the paper is the notion of a "separator" which generalizes Neyman's "generalized distance" and "alternative χ^2" forms. A deeper discussion of separators, as well as a treatment of hypothesis testing, will be given in future papers.

J. Kiefer.

Nandi, H. K. On Type B_1 and Type B regions. Sankhyā 11, 13–22 (1951).

Consider the problem of testing the composite hypothesis $H(\gamma=\gamma_0)$ independent of the parameters $\theta_1, \cdots, \theta_k$, where the probability density function at the sample point $E=(x_1, \cdots, x_n)$ is given by $p(E|\gamma, \theta)$, $\theta=(\theta_1, \cdots, \theta_k)$. Let S denote the set of all critical regions for testing $H(\gamma=\gamma_0)$ which are of constant size α and are unbiased independently of θ. The author obtains sufficient conditions on the form of $p(E|\gamma, \theta)$ for the existence of a critical region of Type B_1 (i.e., a uniformly most powerful region in S) and, in case such a region fails to exist, somewhat weaker conditions are given ensuring the existence of a critical region of Type B (i.e. a locally most powerful region in S). Several examples are given which illustrate the results of the paper.

R. P. Peterson (Seattle, Wash.).

Rao, C. Radhakrishna. A theorem in least squares. Sankhyā 11, 9–12 (1951).

Simpler proof of a result of the author [Sankhyā 7, 237–256 (1946); these Rev. 8, 41]. *B. Epstein.*

Quenouille, M. H. Computational devices in the application of least squares. J. Roy. Statist. Soc. Ser. B. 12, 256–272 (1950).

The purpose of this paper is to extend the techniques customarily used in least squares and the analysis of variance. It is usually assumed that the effects being tested are orthogonal. However, non-orthogonality is commonly encountered in such fields as economics and biology. Consequently, the author gives a number of practical computational methods for treating non-orthogonal data. A number of carefully worked numerical examples illustrate the procedures. *B. Epstein* (Detroit, Mich.).

TOPOLOGY

✓***Appert, Antoine, et Fan, Ky-. Espaces topologiques intermédiaires. Problème de la distanciation.** Actualités Sci. Ind., no. 1121. Hermann & Cie, Paris, 1951. 160 pp.

In an earlier pair of booklets [Propriétés des espaces abstraits les plus généraux . . . , Hermann, Paris, 1934] the first named author has generalized the notion of topology by studying these axioms for a "closure" operator defined in the class of subsets of a set P: $(\mathcal{V})$ $E^-\subset(E\cup F)^-$, (α) $E^{--}=E^-$, (D) $(E\cup F)^-\subset E^-\cup F^-$. Spaces $\mathcal{V}_\alpha$ satisfy the first two; spaces $\mathcal{V}_{\alpha D}$ satisfy all three and constitute the T-spaces of Alexandroff and Hopf. One of the theses of this and the earlier work is that the axiom D is often quite unnecessarily added to $\mathcal{V}$ and α; for example it is possible to define open sets, derived set, interior, and other notions often adopted as primitive, for spaces $\mathcal{V}$ and to state what additional properties these concepts must have in order to insure α or α and D. In the earlier work, the point of departure was the notion of derived set, whereas here it is that of closure. The first chapter is concerned with transformations of axiom systems, with such results as this: P is a $\mathcal{V}_\alpha$ space if and only if P and the void set are open, and the union of any class of open sets is open. The second chapter shows how notions of connectedness, (neighborhood) character, etc., should be defined.

The third chapter, due to K. Fan, is devoted to $\mathcal{V}_{\alpha D}$ spaces which satisfy the axiom of Vietoris or that of Tietze (quasi-regular, resp. quasi-normal spaces). (It will be recalled that regular, resp. normal spaces, in the sense of Alexandroff and Hopf are required to be Hausdorff spaces.) This generalization is then justified by the demonstration of theorems generalizing (in the sense of prefixing "quasi-" before the words "regular" and "normal" in suitable statements of) fundamental results such as Urysohn's Lemma, Tietze's extension theorem, the relation between perfect separability, normality and regularity, and the construction of combinatorially similar open coverings of finite systems of closed sets. The fourth chapter, by K. Fan, deals with numerous approaches to quasi-distance (=pseudo-metric) spaces, generalizing (inasmuch as $d(x, y)=0$ does not necessarily imply $x=y$) results of Chittenden, Frink, Alexandroff and Urysohn, and Niemytzki. There is an extended appendix dealing with the axiomatics of topological and uniform spaces. The references to the literature are quite extensive although far from complete. For example, the monograph of J. Tukey, Convergence and Uniformity in Topology [Princeton Univ. Press, 1940; these Rev. 2, 67], is not mentioned, although it covers much of the ground which is explored in more leisurely and complete fashion in the present work. *R. Arens* (Cambridge, Mass.).

Appert, Antoine. Espaces majorés. C. R. Acad. Sci. Paris 232, 1536–1538 (1951).

After making reference to a number of previous notes on a similar topic [the latest is in the same C. R. 231, 753 (1950); these Rev. 12, 518], the author presents a concept generalizing the "abstract écart" and various other approaches to uniform spaces. Various relations among subsets of a battery of more than twenty properties are indicated.

R. Arens (Cambridge, Mass.).

Michael, Ernest. Topologies on spaces of subsets. Trans. Amer. Math. Soc. 71, 152–182 (1951).

This is an extensive study of spaces of subsets of a topological space X. For the most part (4 below is an interesting exception) the theorems are well known for the compact metric case, and the pathology encountered is perhaps more surprising than the positive results. The finite topology (definitions are below) for 2^X is used almost exclusively. (This may be a serious shortcoming; for example, a topology based on locally finite coverings seems to offer advantages.) A sample of the definitions and results follows: Let T be a T_1 topology for X, let 2^X be the collection of nonvoid closed subsets of X and let $C(X)$ be the collection of nonvoid compact subsets. The finite topology 2^T for 2^X is the coarsest topology such that for each U, open in X, both $\{A: A\epsilon 2^X$ and $A\subset U\}$ and $\{A: A\epsilon 2^X$ and $A\cap U$ nonvoid$\}$ are open. 1. If X is regular the union of the members of a compact subcollection of 2^X (of $C(X)$) is closed (resp. compact). The union of a closed subcollection $\mathfrak{A}$ is closed if the union of each subcollection of $\mathfrak{A}$ belongs to $\mathfrak{A}$. (There are misprints in the statement and proof of theorem 2.5.) 2. If X is compact, or separable, so is 2^X. The space 2^X is regular if and only if X is normal. Roughly speaking, $C(X)$ inherits the separation properties of X. (In 4.4.1 and its proof write $C(X)$ instead of 2^X.) 3. If a function f carries X onto Y and $f^{-1*}(y)=\{x: x\epsilon X$ and $f(x)=y\}$ then f^{-1*} is continuous if and only if f is open and closed. If f is continuous, open and closed, then the quotient topology, on range f^{-1*}, agrees with the relativized finite topology.

4. When X is connected and Hausdorff there is a continuous function f on $C(X)$ to X such that $f(A)\epsilon A$ for each $A\epsilon C(X)$ if and only if there is a continuous linear ordering of X. 5. Let U be a uniform structure on X, $|U|$ the corresponding topology, let 2^U be the uniform structure on 2^X which is the natural generalization of the Hausdorff metric, and let $|2^U|$ be the topology of 2^U. Then $|2^U|=2^{|U|}$ if and only if X is totally bounded and satisfies a "uniform normality" condition. Relativized to $C(X)$, $2^{|U|}=|2^U|$. If $[X, U]$ is complete and metrizable so is $[2^X, 2^U]$. However, there are complete uniform spaces $[X, U]$ such that $[2^X, 2^U]$ is not complete.
J. L. Kelley (New Orleans, La.).

Ursell, H. D., and Young, L. C. Remarks on the theory of prime ends. Mem. Amer. Math. Soc., no. 3, 29 pp. (1951). $1.00.

Les principaux résultats de ce mémoire ont déjà été résumés [Bull. Amer. Math. Soc. 54, 291–292 (1948)]. Il comprend trois parties (A), (B), (C).

(A) Rappel de la théorie des bouts-premiers d'un domaine D simplement connexe d'après Carathéodory, les démonstrations étant rejétées en appendice. Définition des ailes d'un bout-premier et ordre de priorité entre les points d'une même aile. Un ordre circulaire étant établi entre les bouts-premiers de D, soit Γ_0 un chemin dans D définissant un bout-premier β et dont l'ensemble limite est minimum (ensemble des points principaux de β), Γ_1 un chemin définissant β et finalement situé à droite d'un chemin tel que Γ_0; l'ensemble des points limites de Γ_1 est l'aile droite de β. On définit de même l'aile gauche. Les points principaux appartiennent aux deux ailes. Si x, y sont deux points de l'aile droite, on dit que x a priorité sur y si tout chemin Γ_1 qui a y pour point limite a aussi x pour point limite. Un point principal a priorité sur tout point de l'aile droite; de même pour l'aile gauche. Cette notion constitue un pré-ordre dans l'ensemble des points d'une aile.

(B) Relations entre les bouts-premiers d'un domaine D et ceux d'un même domaine ou de l'un de ses complémentaires D'. Deux théorèmes principaux sont démontrés. (1) Toute aile A' d'un bout-premier de D' est contenue dans une aile d'un bout-premier de D. (2) Soit A l'aile droite d'un bout-premier α de D, B l'aile gauche d'un autre bout-premier β de D, E l'ensemble des points appartenant aux bouts-premiers de l'intervalle ouvert α, β. Alors, (i) A et B sont adhérents à E; (ii) x, y étant deux points de A, il existe une suite de continus disjoints contenus dans E, dont les distances à x et y tendent vers zéro, et telle que, si a, b sont des disques de centres x, y, l'ensemble $D\cup a\cup b$ sépare les uns des autres les ensembles C_n-a-b à l'exception d'un nombre fini d'entre eux.

(C) Il s'agit dans cette partie, selon les auteurs, de conséquences d'une extension de l'énoncé élémentaire suivant: Trois maisons ne peuvent être reliées à plus de deux distributions publiques sans que les canalisations ne se croisent. Le théorème principal s'énonce ainsi: Soit C un continu borné, D_n les domaines complémentaires. Nommons maillon sur C (link) un sous-continu de C qui r'encontre au moins trois bouts-premiers d'un ou de plusieurs D_n. Étant donné un ensemble infini M de tels maillons, on peut en extraire un ensemble dénombrable A_i, et lui adjoindre un ensemble dénombrable β_k de bouts-premiers des D_n, de sorte que chaque maillon de M contienne un point d'un A_i ou d'un β_k.

Le mémoire indique en outre des exemples, et, en appendice, sept autres résultats dus à Ursell sans démonstration. Il y aurait peut être lier de rattacher à la partie (B) les résultats anciens d'Urysohn sur les différentes espèces de bouts-premiers [Fund. Math. 6, 229–235 (1924)] ainsi que des théorèmes dus à Wolkenstörfer [Dissertation, München, 1929].
R. de Possel (Alger).

Finzi, Arrigo. Sulle curve invarianti per una trasformazione analitica di una superficie. Rend. Sem. Mat. Univ. Padova 19, 317–323 (1950).

The author constructs an analytic homeomorphism of the torus on itself leaving invariant a curve having a continuous tangent along no sub-arc [cf. G. D. Birkhoff, Collected Mathematical Papers, Amer. Math. Soc., New York, 1950, Vol. II, pp. 418–443].
W. Kaplan (Ann Arbor, Mich.).

Edrei, A. On iteration of mappings of a metric space onto itself. J. London Math. Soc. 26, 96–103 (1951).

Let (X, ρ) be a metric space. The author defines a sequence of (not necessarily continuous) mappings g_n: $X\rightarrow X$ to be a C-sequence provided that if $\epsilon>0$, then there exist $\delta>0$ and $N>0$ such that $x, y\epsilon X$ with $\rho(x, y)<\delta$ and $n>N$ implies $\rho(g_n(x), g_n(y))<\epsilon$. This notion is applied to the iterates of a mapping f: $X\rightarrow X$ to obtain various results; for instance, if X is totally bounded, if $f(X)=X$ and if $f, f^2, \cdots$ is a C-sequence, then f is one-to-one and $[f^n|n=0, \pm 1, \cdots]$ is equi-uniformly continuous. Several examples are constructed.
W. H. Gottschalk (Philadelphia, Pa.).

White, Paul A. On the union of two generalized manifolds. Ann. Scuola Norm. Super. Pisa (3) 4, 231–243 (1950).

The main theorem proved here is that if M_i, $i=1, 2$, is an orientable generalized n-manifold with boundary relative to the space $M=M_1\cup M_2$, and if $M_1\cap M_2$ is the common boundary of M_1 and M_2 and is an orientable generalized closed $(n-1)$-manifold, then M is an orientable generalized closed n-manifold.
E. G. Begle (New Haven, Conn.).

Borsuk, Karol. On an irreducible 2-dimensional absolute retract. Fund. Math. 37, 137–160 (1950).

An example is presented of a 2-dimensional absolute retract such that any 2-dimensional proper closed subset has an infinite 1-dimensional Betti number. In particular, no proper closed 2-dimensional subset is an absolute retract. The construction of this example is too complicated to outline here.
E. G. Begle (New Haven, Conn.).

Hu, Sze-tsen. Cohomology and deformation retracts. Proc. London Math. Soc. (2) 52, 191–219 (1951).

Whitehead's theorem giving necessary and sufficient conditions for a subcomplex L of a complex K to be a deformation retract of K was extended by Hu [Proc. Cambridge Philos. Soc. 43, 314–320 (1947); these Rev. 9, 197] to the case of compact absolute neighborhood retracts. In the present paper the restriction of compactness is removed. In addition, each of the two following sets of conditions is shown to be necessary and sufficient for the absolute neighborhood retract X to be a deformation retract of the absolute neighborhood retract Y:

A. 1) Y is 1-aspherical relative to X.
2) For each $r\geqq 1$, every mapping f: $S^r\rightarrow Y$ is homotopic with a mapping f^*: $S^r\rightarrow X$.
3) For each $r\geqq 1$, any mapping f: $S^r\rightarrow X$ which is null-homotopic in Y is null-homotopic in X.

B. 1) Y is 1-aspherical relative to X.
2) Y is r-simple relative to X for each $r>1$.
3) $H^r(Y \bmod X, \pi^r)=0$ for each $r>1$, where $\pi^r=\pi^r(Y, X)$.

E. G. Begle (New Haven, Conn.).

★**Hirsch, Guy. Quelques relations entre l'homologie dans les espaces fibrés et les classes caractéristiques relatives à un groupe de structure.** Colloque de topologie (espaces fibrés), Bruxelles, 1950, pp. 123–136. Georges Thone, Liège; Masson et Cie., Paris, 1951. 175 Belgian francs; 1225 French francs.

Let M be a fiber bundle with base space m, fiber F, and structural group G acting transitively on F. Let G_0 be an invariant subgroup of G. By identifying the points of F modulo G_0, we fiber F by fibers f with the structural group G_0. The space $\mathbf{M}$ of the fibers f carries two fiber structures, as the base space of a bundle with M as total space and as the total space of a bundle with m as base space. In particular, if F is a sphere and G the orthogonal group, we can take G_0 to be the group generated by the antipodal transformation. Then $\mathbf{M}$ is a bundle of real projective spaces over m. The main idea of the paper is that the study of this bundle will give bundle invariants of the given bundle. For instance, in the example mentioned above, the fiber of the bundle M over $\mathbf{M}$ being a pair of points, there is a one-dimensional characteristic class u (mod 2) in $\mathbf{M}$. If the fiber spheres are of dimension $n-1$, then we have

$$(u)^n = W^1(u)^{n-1} + W^2(u)^{n-2} + \cdots + W^n,$$

where multiplication is by cup product and where W^i, $i=1, \cdots, n$, are the Stiefel-Whitney classes. This formula makes it possible to define the Stiefel-Whitney classes by homology considerations and without the use of the associate bundles. It has been known to Wu Wen-tsun and has been proved by Wu and the reviewer [unpublished] as a consequence of Whitney's duality theorem. The author obtains in this way invariants of general fiber bundles. They are, however, not susceptible of simple geometrical interpretation as in the case of sphere bundles. In his treatment the author makes use of results in his earlier papers.

S. Chern (Chicago, Ill.).

Kudo, Tatsuji. Homological properties of fibre bundles. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **1**, 101–114 (1950).

Let A be a fibre bundle, fibre F, over the polyhedron B. The author establishes connections between the homology groups of A, F and B. [The work is related to that of G. Hirsch [C. R. Acad. Sci. Paris **227**, 1328–1330 (1948); these Rev. **10**, 558] and J. Leray [ibid. **222**, 1366–1368, 1419–1422 (1946); **223**, 395–397, 412–415 (1946); these Rev. **8**, 49, 166].] He finds invariants of the bundle, namely characteristic subgroups of the homology groups of B, with coefficients in the homology groups of F. It is shown in particular that $H(A)$, the homology group of A, is a subgroup of a factor group of $H(B)\otimes H(F)$ [the latter interpreted as $H(B; H(F))$]. The main tools are: (1) The relative homology groups $H^p(A^q, A^r)$ (where A^q means the part of A over the q-skeleton of B), in particular the images in $H^p(A^q, A^{q-1})$ of $H^p(A^q, A^{q-k})$, $k\geqq 1$, and the kernels of the injection of $H^p(A^q, A^{q-1})$ into $H^p(A^{q+k}, q^{q-1})$, $k\geqq 0$, and (2) the fact that $H^p(A^q, A^{q-1})$ can be identified with the q-chain group $C^q(B; H^{p-q}(F))$, and that, for connected structure group, the boundary operator in B coincides with the natural map of $H^p(A^q, A^{q-1})$ into $H^{p-1}(A^{q-1}, A^{q-2})$. As applications, Gysin's results on sphere bundles [Comment. Math. Helv. **14**, 61–122 (1942); these Rev. **3**, 317] are reproved; and two theorems (due to the reviewer for Lie groups [Ann. of Math. (2) **42**, 1091–1137 (1941); these Rev. **3**, 143]) established in general: I. If $H(F)\rightarrow H(A)$ is an isomorphism, then $H(A)\approx H(B)\otimes H(F)$. II. If B is an odd homology-sphere, and F a homogeneous space, then $H(A)=H(B)\otimes H(F)$. The proofs here are of the nature of addition theorems, building up B by addition of simplices. In the course of this a "homology Feldbau lemma" is proved: If B is acyclic, then $H(A)\approx H(F)$. The complete definitions and proofs, though quite natural, are too long to be described; exact sequences are employed throughout. An appendix considers the class of bundles induced by all maps, homotopic to a given one, of B into the base space B^* of a bundle F^*. It is shown that this class is itself a fibre bundle, whose fibre is any of the induced bundles, and whose base is the given homotopy class of maps of B into B^*.

H. Samelson.

Borel, Armand. La transgression dans les espaces fibrés principaux. C. R. Acad. Sci. Paris **232**, 2392–2394 (1951).

The paper is a continuation of two earlier notes of the author [same C. R. **230**, 943–945 (1951); **232**, 1628–1630 (1951); these Rev. **12**, 435, 848] and contains theorems on the cohomology of the fiber and the base space of a universal principal fiber bundle and their relationship under transgression. Let the fiber G be a compact connected Lie group, and let $E(n, G)$ be a principal fiber bundle universal in the dimension n, with the base space $B(n, G)$. Let A be the ring of integers or a field. Suppose that the cohomology ring $H(G, A)$ of G with the coefficient ring A is the exterior algebra of a free A-module having a base $p_1, \cdots, p_r$ formed by the elements of the odd degrees $m_1, \cdots, m_r$. The first theorem of this paper states that, up to dimension n, $H[B(n, G), A]\cong A[q_1, \cdots, q_r]$, where q_i is of degree m_i+1. The second theorem generalizes the theorem of Hopf-Samelson and states that, if G is without torsion, $H(G, A)$, where A is the ring of integers, is the exterior algebra of a group having a base formed by absolutely transgressive elements. Similar results hold for coefficients mod p. For coefficients mod 2 a more precise result between the generators of $H(G, A)$ and $H(B(n, G), A)$ is given.

S. Chern (Chicago, Ill.).

Wallace, A. D. Cohomology groups near a set. Anais Acad. Brasil. Ci. **22**, 217–225 (1950).

In Spanier's formulation of the Alexander cohomology theory, $C^p(X)$ denotes the group of all p-cochains of a space X over an abelian group G, i.e. the additive group of all functions from X^{p+1} into G. Let d denote the coboundary operator and $|c|$ denote the kernel (support) of the cochain c. As usual, let $C^p(X, A)$ denote the group of relative p-cochains and $C_0^p(X)$ denote the p-cochains with empty kernels. For $A\subset X$, let $Q^p(X, A)$ be the subgroup of $C^p(X)$ composed of all those c such that $|c|$ does not meet A. The groups $d^{-1}Q^{p+1}(X, A)$ and $dC^{p-1}(X)+Q^p(X, A)$ are respectively called the group of p-cocycles near A and the group of p-coboundaries near A. The groups $Q^p(X, A)\cap d^{-1}C_0^{p+1}(X)$ and $dQ^{p-1}(X, A)+C_0^p(X)$ are the group of relative p-cocycles of X near A and the group of relative p-coboundaries of X near A. Thus the author defines the pth cohomology group near A and the pth relative cohomology group of X near A. The main theorem of the paper is as follows. If X is compact Hausdorff and A is closed, then the pth cohomology group near A is isomorphic with the pth cohomology group of A and the pth relative cohomology group of X near A is isomorphic with the pth relative cohomology group of X modulo A.

S. T. Hu (Ann Arbor, Mich.).

Blankinship, W. A., and Fox, R. H. Remarks on certain pathological open subsets of 3-space and their fundamental groups. Proc. Amer. Math. Soc. 1, 618–624 (1950).

Let S denote the 3-sphere, C the closed 3-cell in S consisting of the "horned sphere" of J. W. Alexander [Proc. Nat. Acad. Sci. U. S. A. 10, 8–12 (1924)] together with its interior, P the compact 0-dimensional set in S described by L. Antoine [C. R. Acad. Sci. Paris 171, 661–663 (1920)]. Elegant proofs of the following facts are given. (1) The fundamental group $\pi_1(S-C)$ of the complementary domain of C in S is not finitely generated (it is a locally free group whose rank is not finite). (2) $\pi_1(S-P)$ admits a homomorphism onto $\pi_1(S-C)$, and is therefore not finitely generated. The proofs make use of explicit presentations of the groups involved in terms of generators and relations, obtained from suitable graphs in S. Furthermore, the simply connected open subset M of S described by M. H. A. Newman and J. H. C. Whitehead [Quart. J. Math., Oxford Ser. (1) 8, 14–21 (1937)] is investigated by similar methods, and a simplified proof is given for the result that M is not an open 3-cell.

B. Eckmann (Urbana, Ill.).

Pontryagin, L. S. Homotopy classification of the mappings of an $(n+2)$-dimensional sphere on an n-dimensional one. Doklady Akad. Nauk SSSR (N.S.) 70, 957–959 (1950). (Russian)

In dieser Note wird gezeigt, dass für $n\geqq 3$ die $(n+2)$te Homotopiegruppe $\pi_{n+2}(S^n)$ der n-Sphäre S^n die Ordnung 2 hat, d.h. dass es zwei Homotopieklassen von Abbildungen der S^{n+2} in die S^n gibt. Damit wird ein vom Verfasser früher [C. R. (Doklady) Acad. Sci. URSS (N.S.) 19, 361–363 (1938)] formuliertes irrtümliches Ergebnis richtiggestellt. Auf Grund der Sätze von Freudenthal [Compositio Math. 5, 299–314 (1937)] genügt zum Beweis des Satzes der Nachweis einer wesentlichen Abbildung von S^{n+2} auf S^n für $n=3$. Dieser Nachweis wird in folgender Weise erbracht (der Gedankengang ist ausführlich dargelegt, die Beweise einzelner Schritte jedoch nur angedeutet). Es sei f eine analytische Abbildung von S^{n+2} in S^n. Es gibt einen Punkt $b\varepsilon S^n$, für welchen $f^{-1}(b)$ ein System von zueinander fremden geschlossenen orientierbaren Flächen ist, und derart dass die Abbildung f in der Umgebung von F betrachtet in naheliegender Weise eine Abbildung von F in $V_{n+3,n+1}$ (die Mannigfaltigkeit aller Systeme von $n+1$ linear unabhängigen Vektoren im $(n+3)$-dimensionalen Raum) definiert. Ist W ein "Weg" auf F, d.h. ein System von zueinander fremden einfach geschlossenen glatten Kurven, so erhält man hieraus eine Abbildung φ von W in $V_{n+3,n+2}$, somit in die eigentliche orthogonale Gruppe Ω_{n+3} in $n+3$ Variablen. φ definiert ein Element $a(W)$ der Fundamentalgruppe von Ω_{n+3}, also eine ganze Zahl mod. 2. Für $a(W)$ gilt (alle Gleichungen zwischen ganzen Zahlen sind mod. 2 zu verstehen): 1) Ist W, als ganzzahliger Zyklus aufgefasst, homolog 0 auf F, so ist $a(W)$ gleich der Komponentenzahl $b(W)$ von W; somit, wenn $c(W)=a(W)+b(W)$ gesetzt wird, $c(W)=0$. 2) Für 3 Wege W, W_1, W_2 sei im Sinne ganzzahliger Zyklen W_2 homolog zu $W+W_1$; dann gilt

$$a(W_2)=a(W)+a(W_1),$$

und $b(W_2)=b(W)+b(W_1)+I(W, W_1)$, wo I die Schnittzahl bedeutet; somit ist $c(W_2)=c(W)+c(W_1)+I(W, W_1)$. Für homologe Wege W, W_1 gilt also $c(W)=c(W_1)$. Ist $W_1, \cdots, W_q$, $W_1', \cdots, W_q'$ eine kanonische Homologiebasis von F, so wird $\Gamma=\sum_{i=1}^{q} a(W_i)a(W_i')$ gesetzt; diese Zahl (mod. 2) ist von der Basis unabhängig und erweist sich als eine Homotopieinvariante $\Gamma(f)$ der Abbildung f von S^{n+2} in S^n. Zum Beweis der Invarianz bei Deformation von f ist eine geeignete analytische Approximation der Deformation erforderlich. Für eine nullhomotope Abbildung f ist $\Gamma(f)=0$. Für die bekannte Abbildung g von S^5 auf S^3, für welche in der Freudenthalschen Theorie [loc. cit.] die Entscheidung der Wesentlichkeit offen blieb (Einhängung der Faserprojektion von S^3 auf S^2, gefolgt von derselben Projektion und von nochmaliger Einhängung) findet man durch explizite Konstruktion $\Gamma(g)=1$; g ist also wesentlich.

B. Eckmann (Urbana, Ill.).

GEOMETRY

Lense, Josef. Die Winkeldreiteilung des Herrn Sauerbeck. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1950, 107–110 (1951).

The author subjects the simple rule and compass construction to a careful and detailed examination. He is led to the conclusion that the construction is exact for angles of 90° and 180°. For other values of the angle not surpassing 180° the error does not exceed 8′ 12″, and increases rapidly when the angle grows larger than 180°. The reader will appreciate the two accompanying figures.

N. A. Court (Norman, Okla.).

Taylor, D. G. On certain configurations of congruent triangles. II. Math. Gaz. 35, 80–81 (1951).

Supplementing his earlier paper [same Gaz. 31, 270–278 (1947); these Rev. 9, 371] the author considers two congruent triangles, the first being circumscribed about the other, and the point O about which the first can be rotated into the other. This figure leads to a group of six triangles which may be obtained from any one of them by successive rotations of $\frac{1}{3}\pi$ about the point O. The result is illustrated by an attractive figure.

N. A. Court (Norman, Okla.).

Tuchman, Zevulun. Fourth congruence theorem for tetrahedra. Riveon Lematematika 5, 16–22 (1951). (Hebrew. English summary)

Hohenberg, Fritz. Das Apollonische Problem im R_n und seine Verallgemeinerungen. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 159, 63–70 (1950).

Extending his earlier work [Deutsche Math. 7, 78–81 (1942); these Rev. 8, 399], the author finds, in Euclidean n-space, the center and radius of a sphere cutting $n+1$ given spheres at $n+1$ given angles, or cutting $n+2$ given spheres at angles whose cosines are proportional to $n+2$ given numbers.

H. S. M. Coxeter (Toronto, Ont.).

Katsë, Dēm. N. Analytic relations arising from a certain geometric problem. Bull. Soc. Math. Grèce 25, 115–119 (1951). (Greek)

In the plane let $a_1, a_2, \cdots, a_n$ be the vertices of a (not necessarily simple) oriented closed polygon N_1. From a fixed point z lay off vectors zb_i such that zb_i is equal to the vector from a_{i+1} to a_i. Call N_2 the polygon $b_1, b_2, \cdots, b_n$ and construct (with the same z) N_3 from N_2 exactly as N_2 from N_1, etc. The paper determines the coordinates of the vertices of N_k in terms of the coordinate of N_1.

H. Busemann.

ApSimon, Hugh. Two vertex-regular polyhedra. Canadian J. Math. 3, 269–271 (1951).

The author defines a polyhedron as vertex-regular if its faces are equal polygons and its vertex-figures are equal regular polygons. He describes two infinite polyhedra of this kind. *W. T. Tutte* (Toronto, Ont.).

Berman, Gerald. A generalization of the Pappus configuration. Canadian J. Math. 3, 299–303 (1951).

The author considers a configuration K_n defined abstractly as a system of $3n$ points A_i, B_i, C_i $(i=0, 1, \cdots, n-1)$ and n^2 lines (ij). The line (ij) is on just three of the points, namely A_i, B_j and C_k where $i+j+k\equiv 0 \pmod n$. Then K_3 is the Pappus configuration. The author shows that K_n is realizable as a real configuration whose points lie on a plane cubic. *W. T. Tutte* (Toronto, Ont.).

Lannér, Folke. On complexes with transitive groups of automorphisms. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] 11, 71 pp. (1950).

The discrete groups generated by reflections in the bounding hyperplanes of a spherical or Euclidean simplex have already been enumerated [Coxeter, Regular Polytopes, Methuen, London, 1948, pp. 187–196, 297; these Rev. 10, 261]. So also have the analogous groups whose fundamental regions are simplexes in hyperbolic space of $n\leqq 3$ dimensions [Coxeter and Whitrow, Proc. Roy. Soc. London. Ser. A. 201, 417–437 (1950); these Rev. 12, 866]. The author has completed this work by proving that there are no such groups in hyperbolic space of $n>4$ dimensions, and finding five such groups in 4 dimensions. Three of these five are the symmetry groups of the regular honeycombs $\{p, 3, 3, 5\}$ $(p=3, 4, 5)$ [Sommerville, An Introduction to the Geometry of n Dimensions, Methuen, London, 1929, p. 191]. The remaining two (the last items in the author's Tables III and V on p. 53) are new. Moreover, he has succeeded in describing this family of groups in a purely topological manner, thus extending the work of Threlfall [Abh. Math.-Phys. Kl. Sächs. Akad. Wiss. 41, no. 6, 1–59 (1932)] from 2 to n dimensions. *H. S. M. Coxeter* (Toronto, Ont.).

Buerger, M. J. Some new functions of interest in x-ray crystallography. Proc. Nat. Acad. Sci. U. S. A. 36, 376–382 (1950).

In an earlier paper [Acta Cryst. 3, 87–97, 243 (1950); these Rev. 12, 849], the author discussed the relations between a fundamental set of discrete unit points and its vector set. The theory is now extended to sets of weighted points and to density maps. When a vector set is decomposed into identical polygons, each of the polygons has a different weight. If the weighting in the fundamental set is taken as the numbers of electrons in point-atoms, then the weighting in the vector set is that of the weights of the point-atoms in the squared-crystal. If the weighting in the fundamental set is taken as the electron density, then the corresponding weighting in the vector set maps out the Patterson function. A function is derived which maps the electron density from Patterson F^2 data, except that it maps the density as its square, and maps the original image atom as having a weight equal to the Patterson origin weight. Methods of computation are considered. An image-seeking function enabling the location of the atoms in the crystal structure to be mapped is also derived. Similar functions can be developed for any n-gon. The greater the value of n, the more accurately are variations in electron density reproduced, but the distortion is greater and the computation more tedious. *S. Melmore.*

Buerger, M. J. Tables of the characteristics of the vector representations of the 230 space groups. Acta Cryst. 3, 465–471 (1950).

The author's summary is as follows: "All space groups can be distinguished in their vector representations, except that one cannot distinguish between members of the eleven enantiomorphous pairs. This means that the space groups of all crystals can be distinguished in their Patterson syntheses, except that one cannot distinguish between the members of the eleven enantiomorphous pairs. All but eight of the distinguishable space groups can be recognized in Patterson synthesis merely by their symmetry plus the locations of heavy concentrations. Five of these eight can be distinguished by simple qualitative features of the patterns in the concentration loci, while the other three can be distinguished if the data for the Patterson synthesis is on an absolute basis. A table listing the symmetrical concentrations for the 230 space groups is given. These concentrations are also the only possible Harker sections." *S. Melmore* (York).

Niggli, Paul. Vektorendarstellung der 230 Raumgruppen. Acta Cryst. 4, 190 (1951).

The author shews how the linear and planar concentrations tabulated by Buerger [see the preceding review] can be derived from the tables of square matrices he himself has constructed [Acta Cryst. 2, 263–270 (1949); these Rev. 12 523; ibid. 3, 429–433 (1950)]. *S. Melmore* (York).

Hohenberg, Fritz. Zur Geometrie des Funkmessbildes. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 159, 97–111 (1950).

The properties of the "radio range projection" introduced by K. Rinner [Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. 85, 224–232 (1948); these Rev. 11, 200] are extended by the use of a space transformation so that the radio range projection is normal for the transformed points. This permits established principles of projective geometry to be applied systematically in considering the representation of curves and surfaces by the projection. The effect of curvature of the projection surface as for the earth is considered. *N. A. Hall* (Minneapolis, Minn.).

✱Decnop, Gerard Willem. Het complexe elliptische vlak. Het orientatiebegrip in de elementaire meetkunde. [The Complex Elliptic Plane. The Notion of Orientation in Elementary Geometry]. Thesis, University of Amsterdam, Offsetdrukkerij "Excelsior," 's-Gravenhage, 1951. xii+132 pp.

The author describes a self-dual geometry whose elements are the spins (oriented points) and spears (oriented lines) of the elliptic plane. When a line is defined as a pair of opposite spears, we have the double-elliptic plane or sphere. When also a point is defined as a pair of opposite spins (antipodal points of the sphere), we have the ordinary elliptic plane. The locus of spins at a given distance from a fixed spin is called a spin-circle. The envelope of spears making a given angle with a fixed spear is called a spear-circle.

The same concepts can be defined analytically as follows. A homogeneous set of five complex numbers (x, y, z, t, u) satisfying $x^2+y^2+z^2-t^2-u^2=0$ is called a cycle. The polarized form of this equation is the condition for two cycles to touch. A spin is a cycle with $u=0$. A spear is a cycle with $t=0$. A spin-circle is a pair $(x, y, z, t, \pm u)$. A spear-circle is a pair $(x, y, z, \pm t, u)$. A point is a pair of opposite spins,

i.e., a spear-circle with $u=0$. A line is a pair of opposite spears, i.e., a spin-circle with $t=0$.

Following a suggestion of Coolidge [A Treatise on the Circle and the Sphere, Oxford, 1916, p. 407], the author uses such ideas to improve the formulation and proof of Hart's Theorem: From the eight spin-circles that can be drawn to touch three given spin-circles it is possible to choose four which also touch a fourth spin-circle. He investigates the general triangle by means of covariant and contravariant barycentric coordinates, duplicating some of the classical results of spherical trigonometry. But he goes farther, finding elliptic analogues for many special points, lines and curves related to a Euclidean triangle: the Euler line, the Lemoine line, the de Longchamps point, the Gergonne point, the Nagel point, the isodynamic points, the Torricelli points, the Brocard points, the Steiner ellipse, the Simson cubic and the Taylor cubic [F. G. Taylor, Proc. Edinburgh Math. Soc. **34**, 163–175 (1916)]. *H. S. M. Coxeter.*

Leisenring, Kenneth. Area in non-Euclidean geometry. Amer. Math. Monthly **58**, 315–322 (1951).

The basic figure for measuring area in the elliptic plane is a triangle having three right angles; in the Euclidean plane it is a rectangle (which has four right angles). For the hyperbolic plane the author makes analogous use of a polygon having five right angles: the orthopentagon. Like the Euclidean rectangle, this may have various shapes. But all orthopentagons have the same area. (The proof of this "Theorem 4" might well have been more explicit.) The area of an arbitrary triangle is computed by dissecting an orthohexagon (which is the sum of two orthopentagons). This "finite" method replaces Gauss's dissection of a treblyasymptotic triangle, whose sides are infinite.

H. S. M. Coxeter (Toronto, Ont.).

Esser, Martinus. Self-dual postulates for n-dimensional geometry. Duke Math. J. **18**, 475–479 (1951).

In Verfolg der Untersuchungen von Karl Menger [C. R. Acad. Sci. Paris **228**, 1273–1274 (1949); Rep. Math. Colloquium (2) **8**, 81–87 (1948); Duke Math. J. **17**, 1–14 (1950); diese Rev. **10**, 618; **11**, 533]zur Aufstellung eines hinsichtlich der Begriffe Punkt-Ebene in sich dualen Axiomensystems stellt Verf. ein System von 5 Axiomen für den R_n auf, das die primitiven Begriffe Existenz, Punkt, Ebene, vereinigte Lage und den daraus abgeleiteten Begriff des Simplex enthält: ein Simplex (P, p) ist eine Menge von $n+1$ Punkten $P_0, \cdots, P_n$, von $n+1$ Ebenen $p_0, \cdots, p_n$, so dass P_i mit p_j incident ist für $i\neq j$, nicht incident ist für $i=j$. Mit Hilfe eines speziellen Systems von Punkten und Ebenen, als Flach (flat) bezeichnet, wird dann die Gerade abgeleitet: ein Flach ist ein System von Punkten und Ebenen, bei dem jeder Systempunkt mit jeder Systemebene incident ist und jede Ebene durch alle Systempunkte, jeder Punkt auf allen Systemebenen zum System gehört. Zu jedem Flach gehört mindestens ein Simplex (P, p) und eine Zahl i aus $-1\leqq i\leqq n$, so dass das Flach aus allen Ebenen besteht, die gleichzeitig durch P_0 bis P_i gehen und aus allen Punkten, die gleichzeitig auf p_{i+1} bis p_n liegen. Das für alle diese Simplices gleiche Zahlenpaar $(i, n-i-1)$ heisst die Dimension des Flaches. Eine Gerade ist so definiert als ein $(1, n-2)$ dimensionales Flach. *R. Moufang* (Frankfurt a. M.).

Baker, H. F. On non-commutative algebra, and the foundations of projective geometry. Proc. Roy. Soc. London. Ser. A. **205**, 178–191 (1951).

Im 1. Teil wird gezeigt, wie die Grassmannsche Punktrechnung unter Verwendung der Darstellung von 3 kollinearen Punkten U, V, W durch die Bedingung $\lambda U+\mu V+\nu W=0$, wo λ, μ, ν Parameter aus einem Schiefkörper sind, eine einfache Darstellung der Hessenbergschen Verknüpfung am vollständigen Vierseit [Acta Math. **29**, 1–23 (1905)] gestattet, dessen 3 Paare Gegenseiten von einer beliebigen Geraden g in 3 Punktepaaren AA', BB', CC' geschnitten werden. Bezieht man die 6 Punkte auf 2 feste Grundpunkte O, U auf g, so ergibt sich leicht für die Parameter dieser 6 Punkte die Beziehung von Hessenberg; sie bleibt invariant beim Übergang zu neuen O_1, U_1 auf g, wie eine direkte Rechnung zeigt. Nur dann, wenn die Parameter $\lambda, \mu, \cdots$ einem Körper angehören, gibt es auf g ein Punktepaar, das gleichzeitig zu jedem der 3 Paare $AA', \cdots$ harmonisch liegt. Im 2. Teil untersucht Verf. mit Hilfe desselben Calcüls im Anschluss an eine Untersuchung von W. V. Hodge und D. Pedoe [Methods of algebraic geometry, v. I, Cambridge Univ. Press, 1947; diese Rev. **10**, 396] projektive Beziehungen zwischen Geraden g, g' im Raum; es mögen den Punkten $O, U, E=O+U$ auf g die Punkte $O', U', E'=O'+U'$ auf g' entsprechen und allgemein dem Punkt $P+Q$ resp. dem Punkt $P\cdot Q$ der Punkt $P'+Q'$ resp. $P'\cdot Q'$ auf g', dabei ist $P=O+\lambda U$, $Q=O+\mu U$, $P+Q=O+(\lambda+\mu)U$, $P\cdot Q=O+\lambda\mu U$. So erhalten entsprechende Punkte auf g und g' gleiche Parameter. Sind g und g' windschief und durch 2 Perspektivitäten aufeinander bezogen, so lässt sich eine Konstruktion mittels harmonischer Quadrupel im Raum angeben, die zu einem Punkt auf g das Bild auf g' liefert. Ist g'' zu g' und g windschief, so führt die wiederholte Anwendung dieser Konstruktion auf einen räumlichen Schnittpunktsatz, der sich nicht ohne Weitläufigkeit formulieren lässt. *R. Moufang* (Frankfurt a. M.).

Herrmann, Horst. Vollständig entflechtbare Konfigurationen und Desargues-Sätze in projektiven Räumen. Abh. Math. Sem. Univ. Hamburg **17**, 77–90 (1951).

A configuration C in a projective space P_m is extendable (entflechtbar) to a space P_n, $n>m$, if C is the projection in P_m of a configuration C' in P_n such that every intersection of lines of C' goes into a point of C. For example, the Desargues configuration in a plane is extendable to threespace. The author considers the lattice of subspaces in P_n determined by n lines through a point. Projections of these spaces yield extendable configurations whose incidence tables are expressible in terms of binomial coefficients. He obtains various generalized Desargues theorems relating to perspectivities of these binomial configurations.

Marshall Hall (Washington, D. C.).

Derry, Douglas. The duality theorem for curves of order n in n-space. Canadian J. Math. **3**, 159–163 (1951).

C_n denotes, in real projective space, a continuous $1-1$ image of either the projective line or one of its closed segments; it is assumed: (1) No $(n-1)$-dimensional hyperplane cuts C_n in more than n points. (2) The linear k-space $(0\leqq k<n)$ generated by $k+1$ curve points always converges to a linear k-space designated by (k, s) as the $k+1$ points converge to s. The paper contains an elementary proof of P. Scherk's theorem [Časopis Pěst. Mat. Fys. **66**, 172–191 (1937)] asserting that the dual of C_n has properties (1) and (2). The steps are as follows: (T1) The intersection of the spaces $(n-1, s_1)$ and $(n-1, s_2)$, $s_1\neq s_2$, converges to $(n-2, s)$ when s_1 and s_2 tend to s (on C_n). (T2) The spaces $(n-1, s)$ corresponding to the points s of a sub-arc of C_n do not pass through a same point. (T3) No space point is contained in more than n spaces $(n-1, s)$ of C_n. (T4) If a $(k+1)$-uple $s_1, \cdots, s_{k+1}$ on C_n contracts to a point s, the intersec-

tion of the spaces $(n-1, s_1), \cdots, (n-1, s_{k+1})$ converges to $(n-k-1, s)$. (T3) states that the dual of C_n has property (1), (T2) that it has property (2). *C. Y. Pauc.*

Müller, Hans Robert. Die Bewegungsgeometrie auf der Kugel. Monatsh. Math. **55**, 28–42 (1951).

B et B' représentent deux bases orthonormées de l'espace vectoriel euclidien R à trois dimensions, $\mathbf{r}$, $\mathbf{x}$, des vecteurs de R assimilés aux quaternions purs leur correspondant dans B, $\mathbf{r}'$, $\mathbf{x}'$, des vecteurs de R assimilés aux quaternions purs leur correspondant dans B'. La rotation $\mathfrak{R}$ la plus générale de R s'exprime par $\mathbf{x}'=\overline{\mathbf{P}}\mathbf{x}\mathbf{P}$, $\mathbf{P}$ désignant un quaternion de norme unité et $\overline{\mathbf{P}}$ son conjugué. $\mathfrak{R}$ est représentée par le point $(\mathbf{P})=(-\mathbf{P})$ de l'espace elliptique E à trois dimensions où la distance ϑ de $(\mathbf{P})$ à $(\mathbf{Q})$ est définie par $\cos\vartheta=(\mathbf{PQ})$. À une droite D de E correspond la famille des rotations transformant un vecteur fixé $\mathbf{r}$ en un vecteur fixé $\mathbf{r}'$; pour $\|\mathbf{r}\|=\|\mathbf{r}'\|=1$, l'application biunivoque $(\mathbf{r}, \mathbf{r}')\leftrightarrow D$ est la correspondance cinématique étudiée par G. Fubini, J. Hjelmslev et E. Study. L'image elliptique d'une rotation continue de R est une courbe $\mathfrak{C}$: $\mathbf{P}=\mathbf{P}(t)$ si $\mathfrak{R}$ dépend d'un paramètre t, une surface $\mathfrak{S}$: $\mathbf{P}=\mathbf{P}(u, v)$ si $\mathfrak{R}$ dépend de deux paramètres u et v. Utilisant une méthode de W. Blaschke [Hamburger Math. Einzelschr. **34** (1942); ces Rev. **5**, 215], l'auteur étudie les relations entre les propriétés des mouvements $\mathfrak{R}(t)$ et $\mathfrak{R}(u, v)$ et celles de $\mathfrak{C}$ et $\mathfrak{S}$. De nombreux résultats sont lus sur les formules ou obtenus par des calculs très brefs. Nous nous contentons de signaler deux configurations fondamentales. (F1): L'image cinématique droite $\mathbf{r}'=\mathbf{r}'(t)$ de la tangente T à $\mathfrak{C}$ à l'instant t est le centre instantané de rotation sur la sphère unitaire U; $\mathbf{r}=\mathbf{r}(t)=\mathbf{P}\mathbf{r}'\overline{\mathbf{P}}$ est l'image gauche. $\mathbf{r}'(t)$ engendre la roulante du mouvement sur U, $\mathbf{r}(t)$ la roulette en position initiale. (F2): Les images cinématiques d'une tangente T à $\mathfrak{S}$ au point $\mathbf{P}=\mathbf{P}(u, v)$ décrivent, quand T tourne autour de $\mathbf{P}$, deux grands cercles G et G' avec conservation de la longueur d'arc. Les images cinématiques de la normale à $\mathfrak{S}$ en P sont des pôles de G et G'. La correspondance entre ces pôles quand P décrit $\mathfrak{S}$, conserve les aires. À la même correspondance correspond une famille de surfaces parallèles. *C. Y. Pauc* (le Cap).

✓***Verriest, Gustave. Les nombres et les espaces.** Librairie Armand Colin, Paris, 1951. 188 pp. 200 francs.

Chapter I deals with various kinds of number: natural, transfinite, irrational, transcendental and ordinal. It includes careful proofs that almost all real numbers are transcendental, and that the power of the class of functions exceeds the power of the continuum. Chapter II begins with projective geometry and non-Euclidean geometry in the manner of von Staudt and Klein, and continues with a discussion of the relation between geometric space and physical space, including a description of Lemaître's expanding universe. Chapter III, on groups, is a clear introduction to Galois theory and its application to the classical geometrical problems of antiquity. There is a sympathetic sketch of Galois' life, with a full quotation of the nontechnical parts of his famous last letter to Auguste Chevalier. Chapter IV, on modern algebra, describes residue-classes, rings, quaternions, integral domains, fields and ideals. The final chapter is a history of geometry from the Babylonians and Greeks to the modern "algebra of geometry" in which a geometric object P is a point if the only solutions of the equation $P+X=P$ are $X=P$ and $X=V$ (the empty set). Thus Euclid's celebrated definition of a point ("that which has no parts") is vindicated. *H. S. M. Coxeter.*

Convex Domains, Extremal Problems, Integral Geometry

Green, J. W., and Gustin, W. Quasiconvex sets. Canadian J. Math. **2**, 489–507 (1950).

Let Δ be a subset of the real interval $[0, 1]$ containing at least one point interior to $[0, 1]$. For a subset E of a real normed vector space X of finite dimension ν we define the set ΔE as the set of all vectors x of the form $x=\alpha a+(1-\alpha)b$, where $a\epsilon E$, $b\epsilon E$, $\alpha\epsilon\Delta$. We define further by induction $\Delta^{n+1}E=\Delta(\Delta^n E)$ and $\Delta^\omega E=\sum_1^\infty \Delta^n E$. The set Q is called Δ-convex if $\Delta Q=Q$; the intersection of all Δ-convex sets containing E is denoted by $\Delta[E]$. The relation $\Delta[E]=\Delta^\omega E$ is proved. Using the notation $\Delta^*=\Delta[0, 1]$, the authors prove that the Δ^*-convex sets are identical with the Δ-convex sets, the relations $\Delta^*[E]=\Delta[E]$ and $\Delta^{**}=\Delta^*$ hold always, and finally that the relation $\Delta_1^*\supset\Delta_2^*$ is equivalent with the proposition that the Δ_1-convex sets are Δ_2-convex. Every Δ-convex set is dense in its convex hull; if, moreover, its interior is not empty, then it contains the interior of its convex hull. By considering a homogeneous outer measure function defined on subsets of X and the corresponding inner measure function, we get the following results: the set of points x interior to the convex hull of a Δ-convex set E but not belonging to E has zero inner measure if E has positive outer measure and is empty if E has positive inner measure. Results follow concerning connectedness of Δ-convex sets. Historical remarks. *A. Császár* (Budapest).

Green, J. W., and Gustin, W. On the vector sum of continua. Canadian J. Math. **2**, 508–512 (1950).

Let X be a real normed vector space of dimension ν with the basis vectors a_λ $(\lambda=1, \cdots, \nu)$. Let us denote by A the set of vectors $a=\sum_1^\nu \alpha_\lambda a_\lambda$ with integer coefficients α_λ. Consider ν continua Q_λ such that Q_λ contains the vectors 0 and a_λ and let Q denote the vector sum of the continua Q_λ, that is, the set of all vectors $q=\sum_1^\nu q_\lambda$ with $q_\lambda\epsilon Q_\lambda$. The authors prove that Q is a continuum with nonempty interior, X is the vector sum of A and Q and, if μ denotes a measure on the space X invariant under translation, then $\mu(Q)\geqq\mu(P)$ holds, where P is the vector sum of the line segments P_λ joining 0 to a_λ. These results are used in the paper reviewed above. *A. Császár* (Budapest).

✓***Gale, David. Convex polyhedral cones and linear inequalities.** Activity Analysis of Production and Allocation, pp. 287–297. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

The author demonstrates for convex polyhedral cones in Euclidean n-space several simple consequences of Weyl's fundamental theorem [Comment. Math. Helv. **7**, 290–306 (1935); also, Ann. of Math. Studies, no. 24 (1950), pp. 3–18; these Rev. **12**, 352]. Further properties of cones such as Ville's theorem are proved, as are the main theorem of (finite) game theory and other results which are used in other chapters of this volume. The identity of the theories of convex polyhedral cones and systems of homogeneous linear inequalities is emphasized by stating each result in both geometric and algebraic form. *J. Kiefer.*

✓***Gerstenhaber, Murray. Theory of convex polyhedral cones.** Activity Analysis of Production and Allocation, pp. 298–316. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

The author continues the program of the chapter of the previous review with a systematic description of the proper-

ties of convex polyhedral cones. A cone and its relative interior and boundary are characterized in terms of its extreme halflines. Weyl's theorem is proved for cones containing no subspace of dimension >0, and in general (if the cone is not the whole space) by showing that the projection of a cone on the subspace perpendicular to the largest subspace contained in it is of this character. Projections of cones, relationships between cones and subspaces, and relationships between two cones are studied. Define an n-dimensional cone as its own n-facet and for $r<n$ an r-facet as a maximal subcone in the relative boundary of some $(r+1)$-facet. Relationships between facets are studied, and facets are characterized in several ways. The analysis of a cone into the relative interiors of its facets is given. The contents are too lengthy to detail further. *J. Kiefer.*

Fejes Tóth, László. Über das isoperimetrische Problem. I, II. Mat. Lapok 1, 363–383 (1950); 2, 34–45 (1951). (Hungarian. German and Russian summaries)

Eine elementar abgefasste Besprechung über die Entwicklung des isoperimetrischen Problems mit einigen zum Teil neuen Beweisen der isoperimetrischen Ungleichung, sowie ihrer Verschärfungen. *Author's summary.*

Fejes Tóth, L. Covering by dismembered convex discs. Proc. Amer. Math. Soc. 1, 806–812 (1950).

Let S be a set of convex discs $d_1, d_2, \cdots$. Let s_n be the total area of the first n discs, t_n be the area of the largest square which can be covered by the first n discs, and T_n be the area of the smallest square into which the first n discs may be packed (translations, rotations and reflections are permitted in effecting the coverings and packings). The covering and packing economies of S are defined to be

$$e(S)=\liminf t_n/S_n, \quad E(S)=\liminf S_n/T_n.$$

The set S is said to be normal if the radii of the incircles and circumcircles of the discs have a positive lower bound and a finite upper bound. If S is a normal set and k is a positive integer, it is shown that systems S_k' and S_k'' can be obtained by dividing each set of S into k convex pieces, in two different ways, so that

$$e(S_k')\geqq\frac{k+1}{\pi}\sin\frac{\pi}{k+1}, \quad E(S_k'')\geqq\frac{\pi}{2k}\cot\frac{\pi}{2k}.$$

When each disc has a centre of symmetry and k is odd, the division may be effected so that

$$e(S_k')\geqq\frac{k+2}{\pi}\sin\frac{\pi}{k+2}, \quad E(S_k'')\geqq\frac{\pi}{2k+2}\cot\frac{\pi}{2k+2}.$$

The problem is first reduced to the case when all the domains of S are congruent to one another. Then the decompositions are effected by dividing extreme inscribed and circumscribed polygons into convex quadrilaterals and, in the case when the discs are symmetrical, a symmetrical hexagon. *C. A. Rogers* (London).

Rado, R. Some covering theorems. II. Proc. London Math. Soc. (2) 53, 243–267 (1951).

We start by mentioning a characteristic result of part I [same Proc. (2) 51, 232–264 (1949); these Rev. 11, 51] of the present paper: From a set of congruent homothetic cubes in n dimensions of which the area of the union is A there always can be selected a discrete subset having a total area $\geqq 2^{-n}A$. In the present part II selections of more than one discrete subset of an arbitrarily given set of bodies are considered. The paper contains among various very general propositions the following more concrete result: From the above considered set of cubes there always can be selected two discrete subsets of which the area of the union is $\geqq 2^{1-n}A$. The constant 2^{1-n} cannot be replaced by a greater one. *L. Fejes Tóth* (Veszprém).

Schütte, K., und van der Waerden, B. L. Auf welcher Kugel haben 5, 6, 7, 8 oder 9 Punkte mit Mindestabstand Eins Platz? Math. Ann. 123, 96–124 (1951).

The authors compute, for various integers N, the diameter d of the smallest sphere on which N nonoverlapping circles of (straight) diameter 1 can be drawn. When N is large, the circles tend to be inscribed in N regular hexagons of area $\frac{1}{2}\sqrt{3}$ covering a sphere of area πd^2, and we have asymptotically $d^2\sim 3^{\frac{1}{2}}N/2\pi$. By a more refined argument along the same lines, L. Fejes Tóth [Jber. Deutsch. Math. Verein 53, 66–68 (1943); these Rev. 8, 167] found that

$$d^2\geqq 4\bigg/\left(4-\operatorname{cosec}^2\frac{N}{N-2}\frac{\pi}{6}\right),$$

with the equals sign when the circles are inscribed in the faces of a regular N-hedron $\{p, 3\}$, so that $N=12/(6-p)$ and $d^2=4/(4-\operatorname{cosec}^2\pi/p)$ ($p=2, 3, 4, 5$; $N=3, 4, 6, 12$).

The authors have treated the intermediate cases $N=5, 7, 8, 9$ by an ingenious combination of metrical and topological considerations. It is interesting to observe that the solution for $N=5$ is the same as for $N=6$: we cannot do better than inscribe circles in five of the six faces of the unit cube. When $N=8$ the centres of the circles are not the vertices of a cube but of a square antiprism. The figure for $N=9$ has the symmetry of a triangular prism.

The authors suggest tentative solutions for the further values $N=10, 11, 13, 14, 15, 16, 24, 32$, without proving that these are best possible. For instance, when $N=13$ they conjecture $d=2.09\cdots$; but James Gregory, in an unpublished note, asserted that thirteen equal, nonoverlapping spheres can touch another sphere of the same size, which implies $d\leqq 2$. It remains to be seen which result is correct. For $N=24$ they suggest circles drawn round the 24 vertices of the snub cube. This is most likely the best arrangement, as the value $d^2=7.222\cdots$ differs by only about 5% from Fejes Tóth's lower bound $6.911\cdots$. *H. S. M. Coxeter.*

Nöbeling, Georg. Verallgemeinerung eines Satzes von Herrn W. Maak. Abh. Math. Sem. Univ. Hamburg 17, 95–97 (1951).

Given a fixed segment $S(s)$ of length s in the plane E. Let $\mathfrak{K}$ be the family of all the continua K in E that are congruent to a given one of finite length $L(K)$, and let $\dot{K}$ denote the integral-geometric density of the K's. Let $D(S(s), K)\leqq +\infty$ denote the number of points of $S(s)\cap K$ and put

$$D'(S(s), K)=\begin{cases}0 & \text{if } D(S(s), K)<2\\ D(S(s), K) & \text{otherwise.}\end{cases}$$

The author proves (1) $\int_{\mathfrak{K}}D'(S(s), K)\dot{K}=o(s)$. Since from a formula by Poincaré and the author $\int_{\mathfrak{K}}D(S(s), K)\dot{K}=4sL(K)$, (1) implies that there are only few K's which have more than one point in common with $S(s)$. In the case that the K's are rectifiable arcs, (1) was proved by Maak.

[References: 1) Length of a continuum: cf. three papers by Nöbeling, Jber. Deutsch. Math. Verein. 52, 132–160, 189–197 (1942); Monatsh. Math. Phys. 50, 282–287 (1942); these Rev. 5, 114. 2) Density $\dot{K}$: W. Blaschke, Vorlesungen über Integralgeometrie, vol. I, 2nd ed., Teubner, Leipzig-

Berlin, 1936. 3) Poincaré's formula: Nöbeling, Abh. Math. Sem. Hansischen Univ. **15**, 120–126 (1943); these Rev. **7**, 474. 4) Maak's theorem: same Abh. **12**, 163–178 (1937), p. 167.] *P. Scherk* (Saskatoon, Sask.).

Algebraic Geometry

(17-228
8,(1948-50), (1951)

De Sario, Angela. Sulle curve algebriche piane dette "Perle". Ann. Univ. Ferrara. Parte 1. **8**, ~~34 pp. (1951)~~.

Classification and description of curves of the type $y^s = x^s(1 \pm x)^r$, with special attention to those having an axis of symmetry. *R. J. Walker* (Ithaca, N. Y.).

Stubban, John Olav. Note sur les séries paracanoniques d'une courbe algébrique. Norske Vid. Selsk. Forh., Trondheim **23**, 55–57 (1951).

The author shows that if a paracanonic series g_{2p-2}^{p-2} of an algebraic curve of genus p is compounded with an involution, then C is of genus 4, not hyperelliptic, and has exactly two such paracanonic series. *G. B. Huff* (Athens, Ga.).

Edge, W. L. Humbert's plane sextics of genus 5. Proc. Cambridge Philos. Soc. **47**, 483–495 (1951).

G. Humbert [J. École Polytech. Cahier (1) **64**, 123–149 (1894)] studied a twisted curve C^7 of order 7 and genus 5, the locus of points of contact of lines through a fixed point N_0 with twisted cubics through five other fixed points $N_1, \cdots, N_5$. H. F. Baker [. . . Multiply Periodic Functions, Cambridge Univ. Press, 1907, pp. 322–326] obtained the same curve as the apparent contour from N_0 of the Weddle surface with nodes at $N_0, N_1, \cdots, N_5$. A distinctive feature, pointed out by both authors, is that C^7 lies on five elliptic cubic cones (with vertices at $N_1, \cdots, N_5$ respectively) and is bisecant to the generators of each, so that five of its everywhere finite integrals are elliptic, i.e. have only two independent periods.

The purpose of the present paper is to shew that this curve is the same as that whose canonical model is the intersection of three quadrics in four dimensions with a common self polar simplex. He easily shews that the latter curve has five elliptic integrals, and an abelian group of 32 birational self transformations, and deduces from this many elegant geometrical properties of the projections of the canonical curve from a point, chord, and tangent of itself, which were noted by Humbert of his curve and its plane projections, and some of which he found to be sufficient to characterise the latter. Oddly, the author seems to give no proof that the possession by a curve of genus 5 of 5 elliptic integrals is a sufficient condition for all quadrics through the canonical model to have a common self polar simplex, which seems to be needed to establish the identity of the two curves. It is not hard, however, to fill in this gap in a variety of ways. *P. Du Val* (Athens, Ga.).

(8,(1948-50), 99-112 (1951))

Rosina, B. A. Sulla distribuzione dei diametri di una cubica algebrica piana e in particolare sull'ufficio dei diametri mutuamente coniugati nel piano della curva. Ann. Univ. Ferrara. Parte I. ~~8, 16 pp. (1950)~~.

The diameters of a plane cubic are studied for the four possible forms to which Newton shewed that the equation is reducible by affine transformations [Salmon, . . . Higher Plane Curves . . . , 3rd ed., Hodges, Foster, and Figgis, Dublin, 1879, p. 177]. For the general case $xy^2+ey=ax^3+bx^2+cx+d$ the diameters envelope a conic (ellipse, hyperbola, or parabola according as a is positive, negative, or zero) which reduces to a line-pair intersecting in the origin if $b=0$; for the trident and divergent parabola $xy=ax^3+bx^2+cx+d$, $y^2=ax^3+bx^2+cx+d$ the diameters are all parallel to the y-axis; and for the parabola of Wallis $y=ax^3+bx^2+cx+d$ all the diameters coincide in the line through the inflexion parallel to the y-axis. *P. Du Val.*

Tibiletti, Cesarina. Sulle curve intersezioni complete di due superficie. Ann. Mat. Pura Appl. (4) **31**, 69–81 (1950).

Converse theorems of Valentiner and Halphen which were originally proved by unrelated methods are proved here with the aid of simple properties of the resultant of two polynomials and the "$Af+B\varphi$ theorem" of Noether. The theorems in question are: (a) If a twisted curve $\Gamma_{\mu\nu}$, of order $\mu\nu$ is the complete intersection of two generic surfaces of orders μ and ν, then its projection from a generic point onto a plane is a curve of order $\mu\nu$ with $\frac{1}{2}\mu\nu(\mu-1)(\nu-1)$ double points which lie on a curve of order $(\mu-1)(\nu-1)$ [H. Valentiner, Acta Math. **2**, 136–230 (1883)]. (b) A plane curve $L_{\mu\nu}$ of order $\mu\nu$ with $\frac{1}{2}\mu\nu(\mu-1)(\nu-1)$ double points on a curve of order $(\mu-1)(\nu-1)$ is the projection of a twisted curve which is the complete intersection of surfaces of orders μ and ν [G. H. Halphen, J. École Polytech. Cahier (1) **52**, 1–200 (1882)]. *H. J. Muhly* (Iowa City, Iowa).

Baldassarri, Mario. Ricerche su certe classi di superficie d'ordine n dello S_{n-p+1}. Rend. Sem. Mat. Univ. Padova **20**, 167–183 (1951).

Let $|C|$ be the linear system of hyperplane sections of a normal surface F. If $|C|$ contains partially one (or several) linear systems (of freedom >0) of rational curves, denote by $|C'|$ the residual of $|C|$ with respect to such a linear system; if several such systems exist choose $|C'|$ to have minimum genus and (if there are several such systems of equal genus) minimum degree. The system $|C'|$ may be decomposed in a similar way, and we have

$$|C| = |(C-C')+(C'-C'')+\cdots+(C^{(s-1)}-C^{(s)})+\rho|,$$

where $|C-C'|$, $|C'-C''|$, $\cdots$ are effective systems of rational curves, and $|\rho|$, which admits no further reduction, has genus π. If such a decomposition exists, F is said to have residual of genus π. The author's main result is that every regular F^{2p+i} $(i>0)$ in S_{p+i+1} admits a decomposition in this form. These surfaces are all rational, and the proof is based on a study of their plane representation. The author asserts that it is sufficient to consider the case in which $|C|$ is mapped by a linear system with distinct base-points and ordinary singularities, and shows that any such system (of the right degree and freedom) can be reduced by Cremona transformations to one which contains partially a pencil of lines, or to a system of C^9 with eight triple base-points (which is the projective model of a surface of order 9 in S_9 which seems to the reviewer to be a case of exception to the author's theorem). The author points out that "in general" the systems of rational curves $|C-C'|$, $\cdots$ are of freedom 1. He finally proves that a necessary and sufficient condition that $\pi=0$ is that the curves $|C-C'|$ cut on a generic C a linear series which is nonspecial. *J. A. Todd.*

Baldassarri, Mario. Su alcune proprietà proiettive delle superficie d'ordine $2p+1$ dello S_{p+2}, non rigate. Rend. Sem. Mat. Univ. Padova **20**, 184–193 (1951).

The surfaces in question are those whose residual (in the sense explained in the paper reviewed above) is of genus

zero. It is shown that, in general, such surfaces contain a homoloidal net of rational normal C^{p+2}, and at least two rational normal C^p whose containing spaces do not lie in an S_{p+1}. These surfaces can thus be regarded as generalisations of the cubic surface in S_3 ($p=1$). *J. A. Todd.*

Burniat, Pol. Surfaces canoniques quadruples. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 203–207 (1950).

Dans un mémoire précédent [Acad. Roy. Belgique. Bull. Cl. Sci.(5) **32**, 489–507 (1947); ces Rev. **9**, 57] l'auteur a prouvé l'existence des surfaces canoniques quadruples régulières pour toutes les valeurs de p_g. Il expose ici une construction qui permet d'avoir également de telles surfaces ayant une irrégularité q arbitraire. À cet effet, l'auteur considère sur une surface F_0 une involution I_4 composée au moyen d'une involution I_2: si F_1 est l'image de I_2, la courbe de diramation étant Δ_0, I_4 définit sur F_1 une involution I_2' qui a pour image F_2 (courbe de diramation Δ_1). L'auteur établit alors des conditions nécessaires et suffisantes que doivent vérifier Δ_0 et Δ_1 pour que la surface quadruple F_2 soit canonique. Il applique ensuite son procédé au cas où F_0 est rationnelle: par exemple, dans l'espace (x, y, z_0, z_1), si F_0 est le plan (x, y), Δ_0 et Δ_1 les courbes formées respectivement de $2s+2$ et $2r+2$ droites concourantes, la surface F_2: $z_0=\sqrt{\Delta_0}$, $z_1=\sqrt{\Delta_1}$ est canonique, de genre géométrique $p_g=rs$ et d'irrégularité $q=r+s$. Elle représente les couples de points de deux courbes hyperelliptiques de genres respectifs r et s. Du même auteur, sur le même sujet, voir aussi: Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **24**, no. 1602 (1950); ces Rev. **12**, 438. *L. Gauthier* (Nancy).

Burniat, Pol. Sur les surfaces canoniques de genres $p_g=4$, $p^{(1)}\geq 11$. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 880–896 (1950).

Burniat, Pol. Sur les surfaces canoniques de genres $p_g=4$, $p^{(1)}\geq 11$. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 241–251 (1951).

Les surfaces canoniques régulières de genre $p_g=4$ connues jusqu'ici vérifiaient $p^{(1)}\leqq 10$ [Enriques, Le superficie algebriche, Zanichelli, Bologna, 1949, cf. Chap. VIII, pp. 267–284; ces Rev. **11**, 202]. L'auteur met ici en évidence plusieurs familles de surfaces canoniques régulières de genre $p_g=4$ correspondant à des valeurs plus élevées de $p^{(1)}$, obtenues par induction à partir de sous-familles qui sont formées de surfaces doubles canoniques ayant les mêmes genres. (Involution I_2 sur une surface F^n de genre $p_g=0$.) Le mémoire débute par l'étude des conditions que doivent vérifier F^n et la courbe de diramation pour répondre à la question. Ensuite l'auteur examine les cas où F^n appartient à S_3.

La première famille est formée des F^{12} admettant les arêtes d'un tétraèdre T quadruples, et une courbe double C^8 de genre 3 bisécante aux arêtes de T. Cette famille vérifie $p^{(1)}=13$, dépend de 26 modules et contient celle des surfaces F^6 doubles, où F^6 est la surface d'Enriques associée à T. La seconde famille est formée des F^{14} associées à un angle tétraèdre complet T dans lequel un couple d'arêtes opposées a, a' est distingué: Une F^{14} admet a, a' sextuples, les autres arêtes de T quadruples, et une droite double dans le plan de a, a'; elle admet en outre une courbe double C^8 de genre 5 quadrisécante à a, a' et bisécante aux autres arêtes de T. Cette famille correspond à $p^{(1)}=15$ et dépend de 23 modules (le premier mémoire indique 21 modules: cette erreur est corrigée au début du second mémoire). Elle contient celle des surfaces $2F^7$, où F^7 admet a, a' triples et les autres arêtes de T doubles.

La troisième famille est formée des F^{16} associées à un trièdre T d'arêtes n_1, n_2, n_3. Si n_2', n_3' sont les droites infiniment voisines de n_2, n_3 dans les faces n_1n_3, n_1n_2 respectivement, n_4 une droite arbitraire passant par le sommet, n_5 une droite du plan n_2n_3, γ une conique sécante à n_2, n_3, n_2', n_3' et n_4, C^8 une courbe d'ordre 8 de genre 3 bisécante à n_1 et n_5, trisécante à n_2, n_3, n_2', n_3' et quadrisécante à n_4, une surface F^{16} est définie comme possédant n_2, n_3, n_2', n_3' sextuples, n_1, n_4, n_5 quadruples, et γ et C^8 doubles. Cette famille correspond à $p^{(1)}=17$ et dépend de 18 modules; elle contient celle des surfaces $2F^8$, où F^8 est définie par les singularités moitiés. Le second mémoire se termine par l'étude de surfaces doubles $2F^9$ où F^9 est de bigenre 1 à sections de genre 7, dont la classe correspond à $p^{(1)}=19$ et dépend de 12 modules: l'existence d'une sur-famille contenant celle-ci n'est pas démontrée. *L. Gauthier.*

Fano, Gino. Chiarimenti su particolari superficie aventi tutti i generi uguali all'unità. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **84**, 94–96 (1950).

By F^{10} is intended a surface of order 10 in S_6, with all genera equal to unity, whose hyperplane sections are canonical curves of genus 6. The author refers to a remark of Du Val [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **15**, 276–279, 345–347 (1932)] that F^{10} can have a double point in two distinct ways, the projection of F^{10} from its double point being in one case the complete intersection of three quadrics in S_5, in the other a double Veronese surface. He points out that the latter case is determined by either of the following equivalent sufficient conditions: The general hyperplane section of F^{10} is equivalent to a plane quintic (which was the basis of Du Val's approach), or F^{10} has a net of quintics of genus 2 (which is necessarily self-residual with respect to hyperplane sections). It is also pointed out, from a count of parameters, that the most general F^{2p-2} in S_p, whose hyperplane sections are canonical curves of genus p, cannot for $p>23$ have the general canonical curve of genus p as a section. The reviewer was not able to follow this argument, but it seems to him that the correct result is stronger than that stated; the number of projectively distinct surfaces F^{2p-2} is ∞^{19}, and each has ∞^p hyperplane sections; the number of projectively distinct canonical curves which can figure as hyperplane section of a surface of the family is thus at most ∞^{19+p}, and this is not the whole family of birationally distinct curves of this genus if $19+p<3p-3$, i.e. if $p>12$ (instead of $p>23$). *P. Du Val* (Athens, Ga.).

Roth, L. On a class of unirational varieties. Proc. Cambridge Philos. Soc. **47**, 496–503 (1951).

In an earlier note [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **9**, 62–64 (1950); these Rev. **12**, 438] the author has indicated certain extensions of a theorem of Castelnuovo and their applications. In the present paper, these and other extensions are examined in detail. Let V_r ($r\geqq 3$) be an irreducible variety which carries an r-dimensional system Σ of elliptic curves C which invades V_r, and which is such that the curves C which pass through a generic point P of V_r form an irreducible, rational, one-dimensional system $\{C\}$. The system Σ is assumed to be unrestricted (that is, as P moves over a V_k ($2\leqq k<r-1$) the corresponding curves $\{C\}$ generate a V_{k+1}) and free (that is, the passage of a curve C through an assigned point does not entail its passage through any other point). The variety V_r is defined in a field k_0 of characteristic zero, and Σ is defined in an extension k of k_0. Let K denote a pure

transcendental extension of k. Under these conditions one of the following possibilities must occur: (1) V_r is unirational, (2) for some i $(1 \leqq i \leqq r-1)$, V_r contains an $(r-i)$-dimensional system of varieties V_i, unirational in $K(P)$, where P denotes the point by which V_i is defined. Similar results are obtained when V_r carries a more ample system of elliptic curves with like properties and when the system Σ consists of hyperelliptic curves of genus $p>1$. In the elliptic case the proofs hinge upon the following facts: (a) the curves C of Σ carry a rational series of sets of n (>2) points such that the generic point P of C is $(n-1)$-fold for a unique set of the series (the residual point is called the conjugate of P with respect to the series), and (b) the process of taking successive conjugates is nonterminating. Further generalizations are obtained by considering a system of s-dimensional varieties W_s $(1<s \leqq r-1)$ which have properties (a) and (b).

H. T. Muhly (Iowa City, Iowa.).

***Chevalley, Claude. Introduction to the Theory of Algebraic Functions of One Variable.** Mathematical Surveys, No. VI. American Mathematical Society, New York, N. Y., 1951. xi+188 pp. \$4.00.

In this book the author develops systematically the theory of fields R of algebraic functions of one variable over arbitrary fields K of constants (for all practical purposes, K is assumed to be algebraically closed in R). The approach is therefore very general, and the treatment incorporates most of the new ideas and methods which have been introduced into the purely algebraic theory of function fields since the appearance of the classical treatise of Hensel-Landsberg. The manner in which the classical material is developed and adapted to arbitrary fields of constants is novel in many respects, and almost every chapter shows distinct traces of the author's original thinking about the subject. The book is deliberately ageometric in character, as witnessed symbolically by the total absence of the word (and the concept) "algebraic curve" throughout the exposition. The author is concerned exclusively with the properties of the field R relative to the field of constants K ("birational properties", in the terminology of algebraic geometry), and his object is to derive these properties directly from the study of the field R rather than through the medium of projective models of R/K. That this object can be achieved at all is due to an ideal state of affairs which prevails in the theory of algebraic curves and which has no counterpart in the theory of algebraic surfaces or higher varieties. For example, all existent proofs of the Riemann-Roch theorem for algebraic surfaces (and also all attempts, past or present, to extend this theorem to higher varieties) bear not the slightest resemblance to the extremely elegant and intrinsic proof (due to A. Weil and based on the notion of a repartition in R, introduced by the author) which is to be found in this book for the case of dimension 1. For this reason, and at the present state of the theory of algebraic varieties, the book will be somewhat less useful to algebraic geometers than it will be to pure algebraists and, in particular, to those interested in the theory of algebraic numbers. In some instances, a mild concession to geometry could have been made with some advantage (for instance, the intervention of the concepts of simple and absolutely simple points and of normal models would have clarified considerably the material of chapter V dealing with the effect of a ground field extension on a place and on the genus of R/K), but these instances are not very numerous.

After having introduced in chapter I the concepts of a place and a divisor in R/K, the author enters at once, in chapter II, into the very core of the subject: the theorem of Riemann-Roch. For any divisor $\mathfrak{A}$ of R/K one has the vector space $L(\mathfrak{A})$ (over K) of all elements x in R which are divisible by $\mathfrak{A}$. If $d(\mathfrak{A})$ and $l(\mathfrak{A})$ denote respectively the degree of $\mathfrak{A}$ and the dimension of $L(\mathfrak{A})$, the expression $1-l(\mathfrak{A})-d(\mathfrak{A})$ attains a maximum value as $\mathfrak{A}$ runs through the set of all divisors of R/K, and this maximum is defined as the genus g of R/K. A repartition $\mathfrak{x}$ of R is defined as a mapping which assigns to every place $\mathfrak{p}$ of R an element $\mathfrak{x}(\mathfrak{p})$ of R, provided there is only a finite number of places $\mathfrak{p}$ for which $v_\mathfrak{p}(\mathfrak{x}(\mathfrak{p}))<0$ ($v_\mathfrak{p}$ denotes the order function, or the valuation, defined by $\mathfrak{p}$). The repartitions form a vector space $\mathfrak{X}$ over K, and R can be identified with a subspace of $\mathfrak{X}$. Every divisor $\mathfrak{A}$ defines a subspace $\mathfrak{X}(\mathfrak{A})$ of $\mathfrak{X}$ consisting of those repartitions $\mathfrak{x}$ for which $v_\mathfrak{p}(\mathfrak{x}) \geqq m_\mathfrak{p}$, where $m_\mathfrak{p}$ is the exponent of $\mathfrak{p}$ in $\mathfrak{A}$. A differential of R is then defined as a linear function ω on $\mathfrak{X}$ with the property that there exists a divisor $\mathfrak{A}$ such that $\omega=0$ on $\mathfrak{X}(\mathfrak{A})+R$, and when that is so one writes: $\omega \equiv 0 \pmod{\mathfrak{A}}$. In terms of these concepts, the theorem of Riemann-Roch is then formulated as follows: $l(\mathfrak{A})=d(\mathfrak{A}^{-1})-g+1+\delta(\mathfrak{A}^{-1})$, where $\delta(\mathfrak{A}^{-1})$ is the dimension of the space of differentials which are $\equiv 0 \pmod{\mathfrak{A}^{-1}}$. It follows then at once that g is the number of linearly independent differentials of the first kind and that $2g-2$ is the degree of the canonical class. The proofs of these fundamental and most powerful theorems of the theory of algebraic functions of one variable take only about 10 pages. That this can be done at all, without practically any preliminaries, is evidence of the power and economy of any method that can be brought to bear directly on the field R. No similar method is yet discernible in the theory of algebraic functions of more than one variable.

The concepts of local $\mathfrak{p}$-adic theory in R are introduced in chapter III, the notion of a repartition is generalized from R to the $\mathfrak{p}$-adic completions $\hat{R}_\mathfrak{p}$ of R at the various places $\mathfrak{p}$ of R, and the differentials of R are extended to linear functions on the space of the generalized repartitions. In terms of these local concepts it is then possible to define the residue, $\operatorname{res}_\mathfrak{p} \omega$, of a differential ω at a place $\mathfrak{p}$; this turns out to be a certain well defined element of the maximal separable extension Σ_s of K in the residue field Σ of $\mathfrak{p}$. The trace of this element, relative to K, is equal to $\omega(1^\mathfrak{p})$, where $1^\mathfrak{p}$ is the repartition which assigns 1 to $\mathfrak{p}$ and zero to all other places. Using this property of residues, the reader will be able to derive the following meaning of a differential ω, intended as a function on the space of repartitions: let $\mathfrak{A}$ be a divisor such that $\omega \equiv 0 \pmod{\mathfrak{A}}$ and let $m_\mathfrak{p}$ be the exponent of $\mathfrak{p}$ in $\mathfrak{A}$; given a repartition $\mathfrak{x}$, fix for each place $\mathfrak{p}$ a function $x_\mathfrak{p}$ in R such that $v_\mathfrak{p}(\mathfrak{x}-x_\mathfrak{p}) \geqq m_\mathfrak{p}$. Then $\omega(\mathfrak{x})=\Sigma_\mathfrak{p} \operatorname{Tr}_{\Sigma_{s,\mathfrak{p}}/K} \operatorname{res}_\mathfrak{p} \omega x_\mathfrak{p}$. Here $\Sigma_{s,\mathfrak{p}}$ denotes the maximal separable extension of K in the residue field of $\mathfrak{p}$, and $\omega x_\mathfrak{p}$ is intended in a natural sense as a differential (the differentials in R form a vector space also over R). It is to be noted that in the case of nonzero characteristic it may very well happen that $\operatorname{res}_\mathfrak{p} \omega=0$ for a place $\mathfrak{p}$ which is a pole of order 1 for ω. This can happen only if the residue field of $\mathfrak{p}$ is inseparable over K. It is this complication that forces the author to restrict his definition of differentials of the second kind (given in chapter VI) to the case in which R is of characteristic zero. At present it is not yet clear how differentials of the second kind should be defined in the case of nonzero characteristic.

Chapter IV deals with the relationship between two fields R/K and S/L of algebraic functions of one variable, satisfying the conditions $R \subset S$, $K=L \cap R$. An important case is that in which S is a finite extension of R. The main object

is the study of extensions of places from R to S and the setting up of a correspondence between divisors and repartitions in R and in S. This leads to the concepts of the conorm of a divisor in R and the cotrace of a repartition in R, the latter being defined only if S is algebraic over R. If S is a finite extension of R, then it is also possible to define the norm of a divisor in S and the trace of a repartition in S. The properties of traces and norms are then applied to the study of the different $\mathfrak{D}$ of S with respect to R. This is a suitable integral divisor in S and it is defined only under the assumption that S is a finite separable extension of R. If $\mathfrak{P}$ is a place of S and $m(\mathfrak{P})$ is its exponent in $\mathfrak{D}$, then in the classical theory we have always $m(\mathfrak{P})=e(\mathfrak{P})-1$, where $e(\mathfrak{P})$ is the ramification index of $\mathfrak{P}$. It is proved that this equality remains valid if and only if the following conditions are satisfied: (a) the residue field of $\mathfrak{P}$ is a separable extension of the residue field of the place $\mathfrak{p}$ of R which lies below $\mathfrak{P}$; (b) $e(\mathfrak{P})$ is not divisible by the characteristic p. In all other cases, $m(\mathfrak{P})$ is greater than $e(\mathfrak{P})-1$.

Chapter V deals with a special case of the theory developed in the preceding chapter. In this special case, S is obtained from R by an extension $K\rightarrow L$ of the field of constants. It is proved, namely, that if L is a given extension of K, then there exists a field S satisfying the following conditions: (a) $S\supset R$; (b) $S\supset L$; (c) no proper subfield of S satisfies conditions (a) and (b); (d) S is not algebraic over L. Then S is necessarily a field of algebraic functions of one variable over L and is uniquely determined in the following sense: If S' is any other field satisfying conditions (a)–(d), then there exists an isomorphism of S onto S' which maps upon itself every element of R and every element of L. The assumption that K is algebraically closed in R is essential for the various results concerning ground field extensions (these results are known to be true for varieties of any dimension and are related to such geometric concepts as "absolutely irreducible variety", "fields of definition of a variety", etc.). The main object of the chapter is the study of the effect of the extension $K\rightarrow L$ of the field of constants on a place $\mathfrak{p}$ of R and on the genus of R. In the case of characteristic zero the situation is always as follows: (a) every place $\mathfrak{P}$ of S which lies over $\mathfrak{p}$ is unramified with respect to R and (b) the residue field of $\mathfrak{P}$ is generated by L and the residue field of $\mathfrak{p}$. It is shown by examples that if the characteristic is different from zero, then neither (a) nor (b) need be true, even if R is separably generated over K. (If a normal model C of R/K had been used in these examples it would have been apparent that $\mathfrak{p}$ is represented by a point of C which is not absolutely simple. This remark has general validity.) As to the effect on the genus g of R it is shown that the genus g' of S is at most equal to g and that the equality $g=g'$ is true whenever L is separably generated over K. [Also here the use of normal models C/K would have shown that a diminution of the genus is possible only if C carries points which are not absolutely simple. The simplest way of seeing this is to consider the postulation formula for C (i.e., the Hilbert characteristic function of C; if R is separably generated over K, the prime ideal of C/K is absolutely prime and hence the Hilbert characteristic function of C/K is the same as that of C/L) and to observe that this formula yields the genus of C/L if and only if C/L is normal, i.e., if and only if all points of C are also simple with respect to L.]

In chapter VI exact differentials dx are defined ($x\epsilon R$) and the relationship between differentials and derivations in R/K is studied. The last part of the chapter, restricted to fields R of characteristic zero, generalizes a number of classical results concerning differentials of the second kind.

The seventh (and last) chapter is dedicated to the Riemann surfaces S of R in the classical case (K is now the field of complex numbers). The topology of S is defined intrinsically, in terms of the field R itself, and not in terms of any particular choice of an "independent variable" (and certainly not by what the author refers to as the "scissor and glue method"). Since the author finds triangulation a cumbersome procedure, he makes use of the singular homology theory as developed by Eilenberg. A novel feature of this chapter is the manner in which intersection numbers of 1-cycles on S are defined in terms of certain integrals on S. If ω and ω' are differentials of the second kind and P is a point of S, then a number $j_P(\omega,\omega')=-j_P(\omega',\omega)$ can be defined as follows: $j_P(\omega,\omega')=\mathrm{res}_P\, f\omega'$, where f is any function in R such that $v_P(\omega-df)\geqq -v_P(\omega')-1$, with the further specification that if $v_P(\omega)\geqq 0$ then $v_P(f)>0$ [it may be pointed out that $j_P(\omega,\omega')$ can also be expressed as a double integral of $\omega\times\omega'$ over a suitable region in the two-fold symmetric product of S]. We set $j(\omega,\omega')=\sum_P j_P(\omega,\omega')$. Then $j(\omega,\omega')$ is a bilinear function on the product space $C\times C$, where C denotes the vector space of all differentials of the second kind. The important result concerning this bilinear function is the following: (a) $j(\omega,\omega')$ is actually a bilinear function on the product space $C/F\times C/F$, where F is the space of exact differentials; (b) this bilinear function is nondegenerate. As a consequence, given any linear function λ on C/F, there exists in C one and only one coset mod F such that $\lambda(\omega)=j(\omega,\omega')$ for any ω' in that coset and for any ω in C. In particular, if c is a fixed element of the one-dimensional homology group H_1 of S, then $\int_c\omega$, $\omega\epsilon C$, defines a function λ such as above, and it follows then that there exists in C a unique coset $\Omega(c)$ mod F such that $\int_c\omega=j(\omega,\omega')$ for all ω' in $\Omega(c)$ and all ω in C. Now if c' is any other given element of H_1, then the integral $(2\pi\sqrt{-1})^{-1}\int_c\omega'$, $\omega'\epsilon\Omega(c')$, is an integer which depends only on the coset $\Omega(c')$, hence only on c and c'. This integer is the intersection number of c and c'. [The author deals actually with a more general situation. If P and Q are any two mutually disjoint finite subsets of S, he considers the one-dimensional homology group $H_1(S-P,Q)$ of $S-P$ mod Q and defines the intersection number of c and c' for any c in $H_1(S-P,Q)$ and any c' in $H_1(S-Q,P)$.] This definition of intersection numbers, and the theorems on which this definition is based, permit the author to determine the homology group H_1 of S and to prove the existence of a canonical set of 1-cycles (retrosections) on S. Other topics in this chapter are the following: the theorem of Abel; fields of genus 1; the bilinear relations of Riemann. *O. Zariski* (Cambridge, Mass.).

Severi, Francesco. Les images géométriques des idéaux de polynomes. C. R. Acad. Sci. Paris **232**, 2395–2396 (1951).

Let $\mathfrak{J}$ be a homogeneous polynomial ideal and let W be the algebraic variety defined by $\mathfrak{J}$. For every analytical one-dimensional branch γ whose origin is on W and which does not lie on W consider the minimum i of the intersection multiplicities of γ with the various hypersurfaces $\varphi=0$, $\varphi\epsilon\mathfrak{J}$. The set c of all pairs (γ,i) is called the effective behaviour of $\mathfrak{J}$ on W; abbreviation: C.E. ("comportement effectif") of $\mathfrak{J}$ on W. The pair (W,c) is called a base variety. There is then a greatest (complete) homogeneous ideal $\mathfrak{J}(W,c)$ whose C.E. on W is c. The ideal $\mathfrak{J}$ is complete if $\mathfrak{J}=\mathfrak{J}(W,c)$. The base varieties can be regarded as geometric images of homogeneous ideals. [The reviewer was not able to interpret the final statement of the note, to the effect that there is a (1, 1) correspondence between homogeneous ideals and base

varieties. The definitions seem to imply that, for instance, the following two ideals in $k[Y_0, Y_1, Y_2]$ determine the same base variety: $\mathfrak{J}_1=(Y_1^2, Y_2^2)$ and $\mathfrak{J}_2=(Y_1^2, Y_1Y_2, Y_2^2)$.]

O. Zariski (Cambridge, Mass.).

Severi, Francesco. Propriétés des images géométriques des idéaux de polynomes. C. R. Acad. Sci. Paris **233**, 15–17 (1951).

The notations are the same as in the preceding review. The sum and intersection of two base varieties (W, c) and (W', c') are defined as the base varieties associated with the ideals $\mathfrak{J}(W, c) \cap \mathfrak{J}(W', c')$ and $\mathfrak{J}(W, c)+\mathfrak{J}(W', c')$ respectively. A base variety is irreducible if it is not the sum of two base varieties. The following generalization of the theorem of Lasker-E. Noether is stated: Every base variety is a finite sum of irreducible base varieties such that any two of the latter are associated with distinct irreducible varieties. In such a decomposition there may occur also embedded base varieties. If in the decomposition of a given base variety (W, c) one omits the embedded base varieties, one obtains a base variety $(W, \bar{c})$ which coincides locally with (W, c) at each generic point of each irreducible component of W. The proofs will be given elsewhere.

O. Zariski (Cambridge, Mass.).

Fernandez Biarge, Julio. Arithmetical investigation of linear systems of divisors of an algebraic variety. Memorias de Matemática del Instituto "Jorge Juan," no. **11**, 80 pp. (1950). (Spanish)

The author develops the theory of linear systems on an algebraic variety, using on the one hand Artin's concept of "Quasigleichheit" of ideals in integrally closed domains and on the other hand the concept of a derived normal model due to the reviewer. He includes in his treatment the case in which the ground field is not maximally algebraic in the function field of the variety. It is known that in this case a derived normal model V is not arithmetically normal, but that the conductor of the homogeneous coordinate ring R of V is then the prime irrelevant ideal of the ring. This necessitates a slight generalization of the notion of "Quasigleichheit", and this is carried out by the author. The paper includes a systematic treatment of the usual but basic material, such as complete linear systems, virtual divisors, the general divisor of a linear system. The proof of the existence of a complete linear system containing a given effective divisor is reduced very simply to the fact that if U is in an integral ideal in the ring R then the fractional ideal U^{-1} is a finite R-module. Linear systems with assigned base conditions are studied in some detail by methods of valuation theory, and some results are obtained in regard to the law of transformation of such systems under birational correspondences. [A minor correction should be made in the definition of homogeneous ideals given on p. 13. If k is a finite field then the reviewer's theorem referred to at the bottom of that page is not applicable, and one cannot then draw the conclusion that the ideal has a basis consisting of forms. It is this latter property that should be used for the definition of homogeneous ideals.]

O. Zariski.

Dieudonné, Jean. Algebraic homogeneous spaces over fields of characteristic two. Proc. Amer. Math. Soc. **2**, 295–304 (1951).

The results of Chow [Ann. of Math. (2) **50**, 32–67 (1949); these Rev. **10**, 396] on the geometric characterization of the operations of the basic group on a polar system are extended to the case of a basic field of characteristic 2. Let then g be a regular nondefective quadratic form on a $(2r+2)$-dimensional vector space E. The singular subspaces of E are those on which g is zero; their maximal dimension is called the index of g; it will be assumed that this index $s+1$ is $\geqq 3$. If P is the projective space whose points are the one-dimensional subspaces of E, the linear subspaces of P which correspond to singular subspaces of E are called singular. Let T be the set of all singular subspaces of highest dimension s of P. The operations of the basic group on T are those permutations of the set T which are generated by those collineations in P which come from semilinear transformations in E which preserve the form g except for a constant factor and an automorphism of the basic field. The theorem of Chow states that any one-to-one adjacence-preserving operation on T is an operation of the basic group. The proof (for the case of characteristic 2) is given only inasmuch as it differs from Chow's. The given transformation is first extended to the set of all singular spaces of all dimensions from 0 to s; this gives, in particular, a mapping of the set of singular points, and, on every singular space S, this mapping induces a collineation of S onto its transformed singular space; this part of the proof proceeds exactly as in Chow. In order to complete the proof, i.e. to extend our mapping to those points which are not singular, the author constructs inductively a sequence $[s]\subset[s+1]\subset\cdots\subset[s+k]\subset\cdots$ of subspaces of P, beginning with a singular $[s]$ and ending with P, and he extends the mapping step by step to the successive $[s+k]$. This proof could also be carried out in the classical case, giving a somewhat simpler argument than the one used by Chow.

In the case where $s=r$, the system T splits into two disjoint subsystems U and U', exactly as it does in the case of a characteristic $\neq 2$. In order to show this, the author constructs a normal subgroup of index 2 of the orthogonal group of the form g, viz., the group of elements which are representable as products of an even number of transvections. This group operates transitively on the set of singular varieties of dimension s if $s<r$, but T splits into two transitivity classes U and U'. Chow's theorem on the characterization of the basic group of any one of these systems U, U' is still valid in the case of characteristic 2 (provided $r\geqq 4$, as in the classical case).

The reason why the case $r=3$ is exceptional in the classical case is the existence of the triality of Study-Cartan. Making use of spinors, the author shows that the triality still exists in the case of characteristic 2 and accounts for the exceptional character of the case $r=3$.

C. Chevalley.

Weil, André. Arithmetic on algebraic varieties. Ann. of Math. (2) **53**, 412–444 (1951).

The author has extended in his thesis [Acta Math. **52**, 281–315 (1929)] Mordell's theorem (concerning the case $p=1$), proving that any algebraic curve of genus p over any given finite algebraic field k has a finite rank r (minimum number of sets of p points, rational over k, from which all the others can be deduced by means of rational operations). The main tool in proving this deep result is a decomposition theorem (proved in loc. cit., chapter I) extending to arbitrary algebraic varieties a well known elementary result on factorization of rational functions over a curve of genus zero. This theory has been further developed by C. L. Siegel [Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1929**, no. 1 (1930)], A. Weil [Arithmétique et géométrie sur les variétés algébriques, Actualités Sci. Ind., no. 206, Hermann, Paris, 1935] and, very recently, D. G. Northcott [Proc. Cambridge Philos. Soc. **45**, 502–509, 510–518 (1949); these Rev.

11, 390]. In the present work (which is divided into five sections and an appendix), by making explicit some ideal-theoretic concepts implicit in the papers cited above, the proper algebraic foundations for that theory are supplied, and a comprehensive account of its results, including some new ones, is given.

In section I, after having defined specializations, specialization-rings, specialization-ideals, valuation-rings and valuation-ideals, the theorem on the extension of specializations in algebraic geometry is proved, as well as other results including Krull's identification between integrally closed ideals and valuation-ideals. Then the valuation-functions are introduced, and their relations with the divisors of an algebraic variety are investigated. Among other results, the following one is proved. Let V be a nonsingular projective variety defined over a field k; let T be a divisor on V rational over k. Then there are functions x_μ, u_ν on V, defined over k, whose divisors are $(x_\mu)=T+X_\mu-Z$, $(u_\nu)=U_\nu-Z$, where Z is a divisor and the X_μ (as well as the U_ν) are positive divisors with no point in common.

Section II contains a study of absolute values on a field and its subrings, leading to the important notions of distributions (functions conveniently attached to finite sets of elements of a field) and their size. For every point $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_n)$ with algebraic coordinates in a projective space P^n, a height $h(\alpha)$ is intrinsically defined, such that, if $d(\alpha)$ is the degree over the rational field Q of the extension of Q generated by the α_i/α_j, the number of points α in P^n for which $d(\alpha)\leqq d_0$, $h(\alpha)\leqq h_0$, is finite (when n, d_0, h_0 are given). Also the effect on heights of a change of coordinates is studied, and it is shown how, from every theorem on sizes of distributions, a theorem on heights of points can be deduced. Using the remark that a set of $n+1$ functions on V defines a mapping of V into P^n, the following theorem is proved.

Let V be an abstract algebraic variety, complete and normal, defined over an algebraic number-field k (i.e., a finite algebraic extension of Q). Let φ, ψ be two mappings of V into projective spaces, both defined over k; assume that the linear system of primals without fixed components, defined on V by φ and ψ, is both without fixed points and belongs to the same complete system. Then φ, ψ are everywhere defined on V, and there are constants γ, γ', both >0, such that

$$\gamma h[\psi(P)]\leqq h[\varphi(P)]\leqq \gamma' h[\psi(P)]$$

at all absolutely algebraic points P of V.

The following decomposition theorem is then established in section III, together with other related results. Let V be a nonsingular projective variety defined over a field k. Let v be an absolute value, everywhere $<+\infty$, on k. Then to every prime rational divisor W over k on V one can attach a function $\Delta_W(P)$, defined at all the points P of V which are rational over k, taking its values in [0, 1] in such a way that the following properties hold: (a) $\Delta_W(P)$ is 0 if and only if P lies on W, and it is continuous everywhere for the topology defined by v; (b) if z is any function, defined over k on V, with the divisor $(z)=\sum_i m_i W_i$, where the W_i are prime rational divisors over k, then there are constants γ, γ', both >0, such that

$$\gamma \prod_i \Delta_{W_i}(P)^{m_i}\leqq v[z(P)]\leqq \gamma' \prod_i \Delta_{W_i}(P)^{m_i}$$

at all points P, rational over k on V, at which z is defined.

In the case of curves, no distinction has to be made between valuation-functions and divisors, and the whole theory simplifies, as it is shown in section IV. Finally, section V provides substantial motivation for the concept of valuation-functions "attached" to divisors in section I, by dealing with some further general concepts (fractional ideals, centre of a valuation, local ideals and their coherent system), and the appendix concerns divisorial valuations.

B. Segre (Rome).

Habicht, Walter. Topologische Eigenschaften reeller algebraischer Mannigfaltigkeiten. Math. Ann. **122**, 181–204 (1950).

Let $\mathfrak{P}$ be a subset of the n-dimensional projective space S^n over a field K. The author calls $\mathfrak{P}$-varieties the intersections of $\mathfrak{P}$ with algebraic varieties in S^n. (Incidentally, he defines S^n to be the set of all points whose coordinate ratios lie in arbitrary overfields of K, although the consideration of such sets leads to well known logical antinomies.) He proves a few trivial facts about the relationships between these $\mathfrak{P}$-varieties and varieties in S^n, and then turns to the study of the case where $\mathfrak{P}=A^n$ consists of all points whose coordinate ratios lie in some ordered field A. This field has a topology, which defines a topology in A^n. It is shown that every A^n-variety M of dimension $<n$ is closed and nowhere dense. Assume from now on that A is "reel abgeschlossen". Then, given any linear subspace S_0 of A^n and a neighbourhood U of $S_0\cap M$, there exists a neighbourhood V of S_0 in the Grassmann variety which represents the linear spaces of the same dimension as S_0 such that $S\cap M\subset U$ for all S in V. Moreover, if M is irreducible of dimension a, then every point of M which is simple on M has a neighbourhood on M which is homeomorphic to a neighbourhood of a point in A^a. This is established by projecting M on a hypersurface.

C. Chevalley (New York, N. Y.).

Derwidué, L. Le problème de la réduction des singularités d'une variété algébrique. Math. Ann. **123**, 302–330 (1951).

The author presents in this paper an investigation which, he claims, constitutes a complete proof of the famous conjecture that every algebraic variety can be transformed birationally into a variety which is free from singularities. He restricts himself to the case of the complex domain. [In this case, and more generally, in the case of ground fields of characteristic zero, the conjecture has been proved so far only for varieties of dimension $\leqq 3$; for the case of dimension 3 see the reviewer's paper "Reduction of the singularities of algebraic three-dimensional varieties," [Ann. of Math. (2) **45**, 472–542 (1944); these Rev. **6**, 102; this paper will be referred to in this review as RS3].] The author has published a number of other papers on the same subject, in particular a long memoir entitled "Le problème général de la réduction des singularités d'une variété algébrique" [Mém. Soc. Roy. Sci. Liège (4) **9**, no. 2 (1949); these Rev. **11**, 740]. The present paper is described in the introduction as a decisive simplification of the proof given in the above cited memoir. It is stated that the simplification is due chiefly to the author's recent discovery of what he calls "elementary transformations." [These transformations, under the name of "monoidal transformations," have been introduced and fully studied by the reviewer on pages 532–542 of his paper (referred to in the sequel as FBC) "Foundations of a general theory of birational correspondences," [Trans. Amer. Math. Soc. **53**, 490–542 (1943); these Rev. **5**, 11]; they have been extensively used by the reviewer in RS3.] The monoidal (or "elementary") transformations are indeed elementary, in more than one sense, and while

the reviewer has found them a useful tool for the resolution of the singularities, he finds it rather strange that they should have had such a tremendous effect on the author's "proof," for the deeper difficulties of the resolution problem can be affected by such straightforward tools only to a very moderate extent. At any rate, it is stated that his discovery of monoidal transformations has enabled the author to condense "en quelques heures" the 139 pages of his cited memoir to five pages of a note entitled "Méthode simplifiée de réduction des singularités d'une variété algébrique" [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 880–885 (1949); these Rev. **11**, 740]. According to the author, this short note contained the complete solution of the problem, so much so that he felt that "pour moi, le problèmé était bel et bien résolu, mais il me restait à faire admettre ma solution." The present more elaborate version of that note has therefore been written chiefly for the purpose of convincing the unbelievers, and the author acknowledges his indebtedness to van der Waerden who, in a series of conferences with the author (". . . une dizaine de discussions de deux heures chacune, suivies pour chacun de nous de deux ou trois jours de réflection . . ."), has closely and critically scrutinized every single detail of the present "proof" (". . . et il prit dès lors la peine de passer au crible de son esprit critique bien connu les douze pages du texte que je lui apportais, littéralement ligne par ligne").

Before describing and discussing the author's reasoning, a few general remarks about the paper will be in order. Its language is "geometric." The geometric language, when it is not based on a carefully prepared algebraic basis, is never explicit or convincing in algebraic geometry. On this ground alone the author's "proof" could be dismissed as incomplete, for in scientific work it is right to hold every author guilty until he proves himself innocent. However, out of consideration for the importance of the problem, and because of the author's implied belief that his work has not been duly evaluated, this reviewer has reversed his attitude and has read the paper on the assumption that it is the reader who is to be held guilty until he proves himself innocent. In practice that meant that in cases of doubtful points, the reviewer has made an effort to either (a) complete the proof himself or (b) find precisely what is wrong with the proof or (and) (c) find a counterexample. With this approach, the geometric language of the author has the effect of shifting a good deal of the burden from the author to the reader, for in many cases it was at least as difficult to accomplish (a) or (b) or (c) as it was for the author to find his incomplete proofs. However, the problem of resolution of singularities presents not only difficulties of rigor but also difficulties of perception which lie much deeper. It is for this reason that the reviewer was willing to give the author the benefit of the doubt on the formal side of the treatment. He was mainly interested in finding out how the author has dealt with the irreducibly difficult core of the problem. He was forced to conclude that lack of perception of the serious (and not obvious) complications that can (and do) arise in the course of the reduction process have prevented the author from coming to grips with the real difficulties of the problem. And so it happens that the two main pillars of his "proof"—(a) the theorem on first polars on p. 314 ("qui est d'une importance capitale pour la suite") and (b) the "raisonnement fondamental" on p. 317—are represented by statements of which the first is false and the second is far from having been proved (see theorems A′ and B′ below; as a matter of fact also B′, in the general form stated at the bottom of p. 318, is false). As a consequence, the following can be said about the two other statements (see theorems A and B below) which are made by the author and whose logical sum implies the resolution theorem: A has not been proved because its "proof" is based on A′; B has not been proved because its "proof" is based on A and B′. On the whole, the author's contribution merely scratches the surface of things: it belongs to the prenatal phase of the problem, a phase familiar to anybody who has given this problem any thought. (We pass without comment the author's astonishing remark that his "methods" can be extended "without difficulties" to perfect fields of characteristic $p\neq 0$, it being quite superficially clear that they could not possibly be so extended.)

Let V be a k-dimensional irreducible variety in a projective space S_q. To resolve the singularities of V one will begin by applying to S_q a monoidal transformation T whose center is a suitable irreducible nonsingular variety W belonging to the singular variety of V. The effect of T is as follows: (a) it transforms birationally S_q into a nonsingular variety B_q', immersed in some projective space $S'_{n'}$; (b) it transforms birationally V into a variety V' immersed in B_q'; (c) it blows up W into an irreducible $(q-1)$-dimensional subvariety Γ' of B_q'; this variety Γ' is free from singularities and carries a ruling of linear spaces Γ_P' of dimension $q-1-\rho$ (where $\rho=\dim W$) which correspond to the individual points P of W (see FBC and RS3). In a second step of the resolution process, V' and B_q' will replace the original V and S_q, and one will apply to B_q' a monoidal transformation T' whose center is a suitable irreducible nonsingular variety W' belonging to the singular variety of V' (the steps just described are identical to those used in RS3). We shall therefore assume that we are given, to begin with, a nonsingular q-dimensional variety B $(q\geqq k+1)$ containing V and contained in some projective space S_n $(n\geqq q)$. If $k=n-1$ (whence $B=S_n$) it is well known what is meant by a first polar F of V. If $k<n-1$, a first polar of V is defined as a first polar of any hypercone which projects V from an S_{n-k-2}. The set $[F]$ of first polars of V is an irreducible algebraic system (linear, if $k=n-1$); its base locus is the singular variety of V. The trace of $[F]$ in B, i.e., the set of intersections $F\cap B$, is an irreducible algebraic system $[\Phi]$ of $(q-1)$-dimensional subvarieties of B, and also the base locus of $[\Phi]$ is the singular variety of V. If T is a monoidal transformation of B, of the type described above, one defines in an obvious fashion the T-transform $[\Phi']$ of the system $[\Phi]$; this will be then an irreducible algebraic system of $(q-1)$-dimensional subvarieties of B' $(=B_q')$ (since W is a base variety of $[\Phi]$, particular members of $[\Phi']$ may contain Γ' as a component, to a suitable multiplicity). This process is to be repeated as long as the successive transforms V', V'', $\cdots$ of V continue to have singularities. One is thus led to a sequence $\{V^{(i)}, B^{(i)}, [\Phi^{(i)}]\}$. At each stage, the next, $(i+1)$th member of the sequence depends on the choice of the subvariety $W^{(i)}$ of the singular variety of $V^{(i)}$. For the purposes of this review it must be clearly understood that $[\Phi']$ no longer has the same relationship to V' as $[\Phi]$ has to V, i.e., $[\Phi']$ is not the B'-trace of the system of first polars of V' (in relation to the ambient projective space $S'_{n'}$ of B'). The author then proposes to prove the following two theorems: A. Every singular point of $V^{(i)}$ is a base point of $[\Phi^{(i)}]$. B. By a suitable choice of the centers $W^{(i)}$ of the successive monoidal transformations $T^{(i)}$ it is possible to obtain, after a finite number of steps, a transform $V^{(i)}$ of V such that the system $[\Phi^{(i)}]$ has no base points.

It is clear that A and B together would suffice to establish the theorem of resolution of singularities. [The idea of using the first polars and studying their behavior under quadratic transformations (in the case of algebraic surfaces) goes back to B. Levi, and was used in one form or another in every investigation dealing with the resolution problem. It may be noted that theorem B has been proved by the reviewer in the case $q=3$, for arbitrary systems $[\Phi]$ of surfaces; see RS3, Theorem 7′, p. 531.] The essential ingredient of the author's "proof" of A is the following theorem: A′. The generic first polar F of V has an $(s-1)$-fold point at every (proper or improper) s-fold point of V ("improper" means "infinitely near"). Or, in less mysterious language: if a (proper) point $O^{(i)}$ of $V^{(i)}$ is s-fold for $V^{(i)}$, then it is $(s-1)$-fold for the generic $\Phi^{(i)}$. The "proof" of A′ is given only under the following special conditions: $k=n-1$ and the monoidal transformations used in the reduction process are all locally quadratic, i.e., their centers are points. The general case is dismissed in two lines by the mere statement that the proof is similar. Now while we shall see in a moment that A′ is false even under these special conditions, it may be explained now that the general case presents additional difficulties which the author does not perceive. If the center W of T is of positive dimension then it may well happen that V' will contain a generator Γ_P' of Γ' which corresponds to some special point P of W. Whatever conclusions one arrives at in the special case of locally quadratic transformations, these conclusions cannot be automatically used, in the general case, for making statements concerning the behavior of V' at Γ_P'; they only give information about the behavior of V' at subvarieties W' which correspond to the whole of W (and not to proper subvarieties of W). [For a method of dealing with this phenomenon, see RS3.] Furthermore, if $k<n-1$ then the polars Φ are defined in terms of the general projections of V (in its ambient space S_n), and in that case it is not at all clear how the transforms Φ' of Φ are related to the polars of the general projections of V' (in its ambient space $S'_{n'}$) and why this matter should have been dismissed in a few words.

We now come to "theorem" A′ and accept the special conditions under which the author "proves" it. It is not difficult to give counterexamples, even in the simplest case of plane algebraic curves ($k=1$, $q=n=2$), for "theorem" A′ is "almost always" false. Here is an example. Let V be the plane curve $f(x, y)=0$, where $f(x, y)=y^3-x^5$. Then V has a triple point at the origin O and it has a double point at the improper point O' infinitely near O in the direction $y=0$. By the locally quadratic transformation T: $x=x'$, $y=x'y'$, the curve V is transformed into the curve V' given by the equation $x'^2-y'^3=0$, the improper point O' now being represented by the origin $x'=y'=0$. The generic polar Φ of V is $uf_x'+vf_y'+w(xf_x'+yf_y'-5f)=0$, i.e., $3vy^2-5ux^4-2wy^3=0$. The T-transform of Φ is $3vy'^2-5ux'^2-2wx'y'^3=0$, and this has at O' a singular point of multiplicity 2 (and not 1, as is claimed in A′). The error in the author's proof is the following (we use his notations, pp. 313–314): it is taken for granted that if P is generic with respect to G, then G' will be such that also P' will be generic with respect to G'. The fact is that in most cases P' will not be generic with respect to G'. The author commits here the same error that has been committed once before in the "proof" of the following incorrect statement: "the composition of the singularity of a generic projection of an algebraic curve in S_3 is the same as the composition of the corresponding singularity of the space curve itself." The error in this reasoning has been pointed out long ago by the reviewer (see p. 12 of the reviewer's book "Algebraic Surfaces" [Springer, Berlin, 1935]). From the moment that P' is not necessarily generic with respect to G' and hence H_1 is not necessarily a generic first polar of G', the multiplicity of H_1 at O_1' may be greater than s_1-1, the lemma on pencils of hypersurfaces given on p. 313 is then not applicable, and the entire proof on p. 314 breaks down. What is even more serious is that as soon as the generic Φ' has at O_1' a point of multiplicity $\geqq s_1$, then in a second quadratic transformation the fundamental surface Π'' will detach itself at least s_1 times from the transform of Φ' while it will detach itself precisely s_1-1 times from the transform of the generic polar of G'. Thus already after two steps we find that further investigation would be necessary before one can make any assertion concerning the multiplicities of Φ'' at the singular points of G'', as compared with the multiplicities which the first polars of G'' have at these points. What happens after i steps is altogether nebulous.

We now pass to theorem B. Part of the "proof" of this theorem, for dimension k, presupposes theorem A for all dimensions $\leqq k-1$, and hence already on this ground the proof is incomplete. That part of the "proof" which is independent of theorem A is based on a theorem which the author states under the heading: "raisonnement fondamental" (p. 317). We shall explain the gap in the author's proof, and for simplicity we shall consider the case $k=3$, $q=n=4$. We shall also assume for simplicity that the singular variety of V (and hence also the base locus of $[\Phi]$) is of dimension 1. Let $[\Phi_1]$ be the characteristic system of $[\Phi]$, i.e., the systems of surfaces which are intersections of pairs of Φ's. We shall also assume that the characteristic system $[\Phi_2]$ of $[\Phi_1]$ (from which the base curves of $[\Phi_1]$ have been deleted) has no base points (in the notation of the author, we have assumed $\sigma=0$, $k_e=1$). Let γ be an irreducible component of the base curve of $[\Phi_1]$. We assume that γ is free from singularities. We take now the curve γ as the center of our monoidal transformation T and we denote by $[\Phi_1']$ the T-transform of the system of surfaces $[\Phi_1]$. The "raisonnement fondamental," in this special case, is the following assertion: B′. Let γ' be an irreducible curve in Γ' which belongs to the base curve of the system $[\Phi_1']$. If the curve γ is s-fold for the generic Φ and t-fold for the generic Φ_1 and if m is the intersection multiplicity of Φ and Φ_1 at γ, then the intersection multiplicity at γ' of a generic Φ' and a generic Φ_1' is $\leqq m-st$ (whence, at any rate, $<m$).

In the proof the author takes it for granted that γ' meets each generator Γ_P' of Γ' ($P\epsilon\gamma$). In other words, he assumes that γ' corresponds to the whole of γ. He does not consider the possibility that γ' may correspond to some special point P of γ (in which case $\gamma'\subset\Gamma_P'$ and γ' does not meet Γ_Q' if $Q\neq P$). To such a curve γ' the author's reasoning is not applicable. The reviewer has asked himself the following question: May it be that the presence of a base curve γ' contained in Γ_P' implies that P is a base point of $[\Phi_2]$ (contrary to the assumption that $[\Phi_2]$ has no base points)? If that is so, it must be proved. The reviewer doubts that that is so. He knows that the above possibility can actually arise in the case of a linear system of hypersurfaces $[\Phi]$ in S_4 whose successive characteristic systems $[\Phi_1]$ and $[\Phi_2]$ satisfy all the conditions (as to their base varieties) stated above; and he does not see why the fact that we are dealing with polar hypersurfaces should make any difference. At any rate, the author correctly remarks that his reasoning makes no use of any special properties of the polars Φ and that even the fact that the Φ's are complete intersections plays

no role in the proof. He therefore reformulates his "raisonnement fondamental" as a general theorem concerning arbitrary pairs of varieties Φ and Φ_1 in B, of dimensions $q-1$ and q' respectively, $q' \leqq q-1$ (see end of p. 318). Now this generalization is definitely false, even if Φ and Φ_1 are complete intersections. Consider, for instance, the following case: Φ and Φ_1 are respectively a hypersurface and a surface in S_4, having a common point P which is an ordinary n-fold point of Φ and an ordinary ν-fold point of Φ_1 ($n>1$, $\nu>1$); we also assume that Φ and Φ_1 have in common a line γ through P and that the intersection multiplicity at γ is 1 (whence $m=s=t=1$). If γ is taken as the center of our monoidal transformation T, then the following facts can be established in a straightforward manner: (a) the plane Γ_P' belongs to the three-dimensional variety Φ', the latter passing through Γ_P' with multiplicity $\geqq n-1$; (b) Γ_P' meets Φ_1' in a curve γ' which is at least $(\nu-1)$-fold for Φ_1'. It follows that the intersection multiplicity of Φ' and Φ_1' at γ' is $\geqq (n-1)(\nu-1) \geqq 1 > 0 = m-st$. This example shows in fact that the intersection multiplicity of Φ' and Φ_1' at a curve γ' belonging to a generator Γ_P' of Γ' may have no particular relationship to either $m-st$ or m. *O. Zariski.*

Differential Geometry

Mirguet, Jean. Sur une classe de surfaces convexes, définie par le biparatingent. C. R. Acad. Sci. Paris **232**, 1632–1634 (1951).

E désigne un ensemble de l'espace euclidien E_3, M un point d'accumulation de E. La famille (éventuellement vide) des limites de directions de plans définis par les triplets ABC de points de E voisins de M, non linéaires et tels que le rayon du cercle $\mathfrak{C}(A, B, C)$ tende vers zéro, est désignée par "biptg$_0$" en M; il s'agit d'un sous-ensemble du biparatingent en M, défini sans la condition de courbure des triplets. Théorèmes: (1, 1) Une "orthocourbe" (élément lipschitzien de courbe) dont le biptg$_0$ est partout vide, possède en tout point une tangente qui varie continuement. (1, 2) Une "orthosurface" (élément lipschitzien de surface), dont le biptg$_0$ est partout fini, possède en chaque point un plan tangent qui varie continuement. (2) S désignant une orthosurface à paratingent supérieur fini [Revue Sci. **85**, 67–72 (1947); ces Rev. **9**, 18], A l'extérieur de l'ensemble des points où S possède la véritable double courbure [loc. cit.], si en un point M de A le biptg$_0$ n'est pas complet, M est intérieur à un domaine convexe de A. La note est principalement consacrée à la démonstration du Th. 2.

C. Pauc (le Cap).

Bouligand, Georges. Sur les transformations de contact. C. R. Acad. Sci. Paris **232**, 1791–1792 (1951).

Théorème: Une transformation T d'élément plan de contact en point, définie par $X=f(x, y, y')$, $Y=g(x, y, y')$, où f et g ont des dérivées premières, fonctions continues de l'élément (x, y, y'), fait correspondre à un arc A doué d'une tangente variant continuement, un continu Γ sans point intérieur. Idée de la démonstration: Γ est envisagé comme une union finie d'images d'arcs $p=f'(x)$, où $y=f(x)$ représente un sous-arc de A, par une transformation à dérivées continues par rapport à x et p. Pour appliquer la même idée au cas de l'espace E_3, il faudrait montrer que pour une surface $z=f(x, y)$ à plan tangent continu, l'associée $p=p(x, y)$, $q=q(x, y)$ dans E_4 est de mesure tri-dimensionnelle nulle au sens de Carathèodory. L'auteur ne résout pas la question soulevée mais suggère, en cas de réponse négative, d'étendre à E_3 le Théorème en demandant que T soit une transformation de contact. *C. Pauc* (le Cap).

★Raševskiĭ, P. K. Kurs differencial'noĭ geometrii. [A Course of Differential Geometry]. 3d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 428 pp.

The first three chapters of this book (pp. 1–148) deal with plane curves. Stress is laid on the discussion of singular points and asymptotes and on the theory of contact. We also find here the theory of evolutes and involutes, as well as the natural equations in the form $k=\varphi(s)$. The fourth chapter (pp. 149–216) brings the theory of space curves, again with a careful discussion of contact. There we find the Frenet formulas and their usual applications, the natural equations and the fundamental theorem. The fifth chapter (pp. 217–293) contains the elementary theory of surfaces with Meusnier's and Euler's theorems, the lines of curvature and the asymptotic lines. The sixth chapter (pp. 294–340) deals with ruled surfaces, including developables, ending with conjugate nets (Koenigs' theorem). Chapter VII finally brings the intrinsic theory of surfaces, Gauss' and Codazzi's theorems, parallel displacement, the geodesic curvature, geodesics, the bending of surfaces, surfaces of constant curvature and the Gauss-Bonnet theorem. The author uses vector methods throughout; his treatment of the material is quite careful. The book closes with a short historical survey, in which a sketch is given of the work of the Moscow school, to which belonged K. M. Peterson (1828–1861), D. F. Egorov (1869–1930), and B. K. Mlodzeevskiĭ (1858–1923). Peterson, a pupil of Minding, anticipated some of the work of Codazzi, Bonnet, and Schwarz. The Codazzi formulas are here called the Peterson-Codazzi formulas.

D. J. Struik (Cambridge, Mass.).

Giannopoulos, Alex. I. Study of curves in reference to their Mayer (M_r) trihedron. Bull. Soc. Math. Grèce **25**, 82–103 (1951). (Greek. English summary)

Let C be a curve in E^3 and S its osculating sphere at a given point p. The Mayer trihedron M_r is formed by unit vectors t in the direction of the tangent of C, a vector n_2 normal to t in the tangent plane of S at p and a vector n_3 in the direction of the radius of S. Mayer [Bul. Fac. Şti. Cernăuţi **2**, 208–228 (1928)] gave without proof the following analog to the Frenet formulae:

$$\dot{t}=\frac{n_2}{\mathrm{P}}-\frac{n_3}{r}, \quad \dot{n}_2=-\frac{t}{\mathrm{P}}-\frac{n_3}{r}, \quad \dot{n}_3=\frac{t}{r}+\frac{n_2}{\mathrm{T}},$$

where r is the radius of S. These formulae are proved. Then formulae, corresponding to those of Darboux for the usual trihedron, for the cinematics of a rigid body are derived in terms of the Mayer trihedron. Curves with common n_2 (analogous to Bertrand curves with common principal normal) are discussed, and it is shown that also in this case a linear relation between P^{-1} and T^{-1} is characteristic. Also, a relation between the torsions T^{-1} similar to the classical one by Shell hold for a pair of such curves. Finally, curves with common n_3 are treated; again the results are quite similar to those discovered by Mannheim for curves with common binormals. *H. Busemann.*

★Cenov, Iv. Vektorni funkcii na edno promenlivo nezavisimo i tyahnoto priloženie v krivite linii. [Vector Functions of One Independent Variable and Their Application to Curves]. Bulgarska Akademiya na Naukite, Sofia, 1947. 148 pp.

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Marussi, Antonio. Geodesia intrinseca. Ann. Triestini. Sez. 2. (4) **3(19)** (1949), 125–137 (1950).

Mineo, Massimo. Sul trasporto delle coordinate e dell'azimut lungo una geodetica e sul problema inverso sopra una superficie qualunque. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 142–145 (1951).

Gougenheim, André. Transformations des projections conformes de la sphère. C. R. Acad. Sci. Paris **233**, 226–228 (1951).

Schneidt, Max. Über eine spezielle Form der Differentialgleichung aller Flächen eines gegebenen Linienelements. Arch. Math. **2**, 367–374 (1950).

All surfaces on R_3 with a given linear element

$$ds^2=Edu^2+2(EG)^{\frac{1}{2}}\cos\omega du dv+Gdv^2$$

can for $n\neq 0$ and either $E\neq 0$ or $G\neq 0$ be determined by means of the differential equation

$$\frac{\partial}{\partial v}\frac{1}{n}\{E^{\frac{1}{2}}\sin\alpha(K(EG)^{\frac{1}{2}}\sin\omega)+m(-A+\beta_u)\}$$
$$-\frac{\partial}{\partial u}\frac{1}{n}\{G^{\frac{1}{2}}\sin\beta(K(EG)^{\frac{1}{2}}\sin\omega)+m(B+\alpha_v)\}=0,$$
$$m=\frac{\partial}{\partial v}E^{\frac{1}{2}}\sin\alpha-\frac{\partial}{\partial v}G^{\frac{1}{2}}\sin\beta,$$
$$n=\frac{\partial}{\partial v}E^{\frac{1}{2}}\cos\alpha-\frac{\partial}{\partial u}G^{\frac{1}{2}}\cos\beta,$$
$$\omega=\alpha-\beta.$$

From every pair of solutions α, β a surface with the given linear element can be derived by quadratures only. The special case $E=0$, $G=0$ is quite easy. The case $n=0$ leads to a special case of surfaces already known to Goursat [Amer. J. Math. **14**, 1–8 (1892)]. As a special case the author considers two systems of surfaces isometrical with respect to each other and to the rotational surfaces. The last section gives expression for M, L, N and K.
J. A. Schouten (Epe).

Scherrer, W. Stützfunktion und Radius. II. Comment. Math. Helv. **25**, 11–25 (1951).

The author has developed in part I of this paper [Comment. Math. Helv. **20**, 366–381 (1947); these Rev. **9**, 464] the theory of surfaces in ordinary space by use of what might be called a supporting trihedron. The integrability conditions are there formulated in three ways: each one stating two relations involving the radius function, the supporting function, a suitable curvature, and the Beltrami operators with respect to one of the three relative (normalized) fundamental forms of the surface. The following general problem is then considered: Which of the above quantities, together with a surface strip, suffice to determine the surface in a neighborhood of the strip? Two such problems associated with the second fundamental form are here solved in detail. (1) The Gaussian curvature and the relative second fundamental form are sufficient data if the given strip is nonasymptotic with respect to this form. (2) The Gaussian curvature, the radius function, and the supporting function are sufficient data (it is assumed that the radius and supporting functions can be taken as surface parameters) if the strip is nonasymptotic with respect to a certain form (not explicitly mentioned, namely $2a\dot{u}+2b\dot{v}$) depending only on the data and which turns out in solution to be the relative second fundamental form on the strip. The method of proof is to ascertain the relative second fundamental form through the guidance of certain invariance properties of the negative polar reciprocal transformation (here called normal inversion), then to use (1).
W. Gustin.

Finsterwalder, Sebastian. Streifengeometrie. I. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1950**, 209–231 (1951).

A geometric description of a good part of the differential geometry of surface strips. The author starts with the orthogonal axes determined at each point of the middle curve of a strip by the tangent to the middle curve, the normal to the surface strip, and the direction orthogonal to these, and with the related Darboux-Cesàro curvature vector $\mathfrak{F}$ which determines the surface strip except for its location in space. Then the helical surface strips ($\mathfrak{F}$ constant) are presented. Special cases where one or more of the components of $\mathfrak{F}$ vanish are pictured first; the general case follows. There are also sections on equiangular pencils of helical surface strips, on the representation of helical surface strips by points in three-space (Polbild), and on the description of a helical surface strip of finite length by means of a pair of circles on a fixed sphere (Kugelriss).

General surface strips are treated next. There is again a representation by points in three-space (Polbild) and a description by pairs of circles on a fixed unit sphere (Kugelriss), and, in addition, a description by points on a unit sphere (Polriss) and a description obtained by rolling out the polar developable of the middle curve of the strip (Wickelriss). A mechanical relationship between the two descriptions which use spheres is given, and equiangular pencils of surface strips are discussed. Various special cases are considered, including among others those surface strips for which some of the components of $\mathfrak{F}$ vanish, those for which $\mathfrak{F}$ has constant direction, those for which $\mathfrak{F}$ has constant magnitude, and those surface strips which are tangent to cylinders along their middle curves. The author concludes with a discussion of parallel surface strips. *A. Schwartz.*

Kasner, Edward, and De Cicco, John. Theory of turns and slides upon a surface. Proc. Nat. Acad. Sci. U. S. A. **37**, 224–225 (1951).

A turn T_α upon a surface Σ is a lineal element transformation whereby each lineal element is rotated about its point through a constant angle α; a slide S_k upon Σ is a lineal element transformation whereby to each lineal element e of Σ there corresponds a lineal element E which is tangent to the geodesic determined by e, and which is such that the geodesic distance between the points of the elements is the constant k. The commutator of the symbols of an infinitesimal turn and an infinitesimal slide is the symbol of an infinitesimal element transformation called an infinitesimal dilatation. The following theorem is stated in this note: The infinitesimal turns, slides, and dilatations upon a surface Σ

generate a three-parameter group if, and only if, Σ is of constant Gaussian curvature. *L. A. MacColl.*

Blaschke, Wilhelm. Sulla geometria dei tessuti. Archimede 3, 89–97 (1951).

Expository paper.

Alt, Wilhelm. Über die topologische Struktur der Liouvilleschen Netze im Kleinen. Math. Nachr. 5, 161–172 (1951).

The author studies the singular points of a Liouville net on an analytic Liouville surface. He proves that such singularities are isolated, and that in the neighborhood of such a singularity the net is topologically equivalent to either the system of confocal coaxial parabolas in the plane, or the polar coordinate net in the plane. *S. B. Myers.*

✱Charrueau, André. Sur des congruences de droites ou de courbes et sur une transformation de contact liée à ces congruences. Mémor. Sci. Math., no. 115. Gauthier-Villars, Paris, 1950. 72 pp. 500 francs.

Dans ce mémoire l'auteur fait l'étude de certains couples de congruences de droites ou de courbes. Les résultats sont dû à la transformation de contact déterminée par deux points O_1, O_2 et par deux vecteurs libres $\vec{r}_1$, $\vec{r}_2$. Dans l'étude intervient un système M de vecteurs constitué par un couple de moment égal à $\overrightarrow{O_1O_2}$ et par deux vecteurs glissants équipollents à $-\vec{r}_1$, $\vec{r}_2$ et dont les supports passent respectivement par O_1 et O_2. Les trois parties du mémoire sont dédiées aux trois cas qui se présentent: 1) M est le système général, 2) O_1 et O_2 sont confondus, M est formé par un vecteur unique, 3) O_1 et O_2 sont distincts, M est réductible à un vecteur unique. Les résultats d'étude ont été déjà mentionés dans Math. Rev. parceque le travail est un ensemble de résultats publiés par l'auteur [Bull. Sci. Math. (2) **70**, 127–148 (1946); C. R. Acad. Sci. Paris **221**, 274–276, 764 (1945); **225**, 620–622, 792–794, 832, 1055–1058, 1262–1264, 1396 (1947); **226**, 155–157, 364 (1948); **227**, 712–714 (1948); **228**, 359–360, 803–805, 894–896, 1076 (1949); **229**, 334–336, 608 (1949); ces Rev. **8**, 531; **7**, 262; **9**, 158, 200, 305, 466; **10**, 326, 624, 625; **11**, 209]. *F. Vyčichlo* (Prague).

Phlōra, Milt. On families of geodesic parallels. Bull. Soc. Math. Grèce 25, 164–166 (1951). (Greek)

If (on a surface in E^3) in a family of geodesic parallels each curve cuts the geodesic of a one-parameter family of geodesics under the same angle, then the surface is intrinsically isometric to a surface of revolution. *H. Busemann* (Auckland).

Wrona, Włodzimierz. Sur les multivecteurs dans V_n. III. Časopis Pěst. Mat. Fys. 74 (1949), 273–280 (1950). (Polish. French summary)

L'auteur généralise au cas de m-vecteurs d'un espace riemannien V_n à n dimensions certains résultats établis antérieurement dans le cas où $n=2m$ [Nederl. Akad. Wetensch., Proc. **51**, 1291–1301 = Indagationes Math. **10**, 435–445 (1948); ces Rev. **10**, 479]; $a_{\lambda\mu}$ désignant le tenseur métrique de V_n, $K_{\lambda\mu\nu\rho}$ son tenseur de courbure et $\chi=(n(n-1))^{-1}K_{\lambda\mu\nu\rho}a^{\lambda\rho}a^{\mu\nu}$ une courbure riemannienne scalaire de V_n, l'auteur introduit le tenseur

$$U_{\lambda\mu\nu\rho}=K_{\lambda\mu\nu\rho}+2\chi a_{[\lambda[\nu}a_{\mu]\rho]}.$$

Si α, β sont des indices pouvant prendre C_n^m valeurs, un m-vecteur sera désigné par f^α et on associe au tenseur métrique et au tenseur U les tenseurs $\overset{(m)}{a}_{\alpha\beta}$, $\overset{(m)}{U}_{\alpha\beta}$, définis, avec les notations de Schouten, par

$$\overset{m}{a}_{\alpha\beta}\rightarrow\overset{m}{a}_{\lambda_1\cdots\lambda_m\mu_1\cdots\mu_m}=m!a_{[\lambda_1[\mu_1}a_{\lambda_2\mu_2}\cdots a_{\lambda_m]\mu_m]},$$
$$\overset{m}{U}_{\alpha\beta}\rightarrow\overset{m}{U}_{\lambda_1\cdots\lambda_m\mu_1\cdots\mu_m}=[m!/2]U_{[\lambda_1\lambda_2[\mu_1\mu_2}a_{\lambda_3\mu_3}\cdots a_{\lambda_m]\mu_m]}.$$

À tout m-vecteur f^α, l'auteur associe le scalaire

$$\overset{m}{\omega}=-\overset{m}{U}_{\alpha\beta}f^\alpha f^\beta/\overset{m}{a}_{\alpha\beta}f^\alpha f^\beta$$

appelé déviation de ce m-vecteur. Lorsque f^α est simple, $\overset{m}{\omega}$ est égal à la différence $(\overset{m}{\chi}-\chi)$, où $\overset{m}{\chi}$ désigne la courbure associée au m-plan déterminé par le m-vecteur simple considéré. Les m-vecteurs propres de la matrice $(\overset{m}{U}_{\alpha\beta})$ par rapport à la matrice $(\overset{m}{a}_{\alpha\beta})$ sont dits les m-vecteurs principaux de V_n. Les propriétés de ces m-vecteurs, par $m=p$ et $m=n-p$, sont étudiées par dualité ainsi que le cas des espaces d'Einstein et des espaces conformes à un espace euclidien. *A. Lichnerowicz* (Paris).

Pinl, M. J. Geodesic coordinates and rest systems for general linear connections. Duke Math. J. 18, 557–562 (1951).

Let coordinates in L_n be chosen such that $\Gamma^\kappa{}_{(\mu\lambda)}=0$ in $\underset{0}{\xi}^\kappa$. Then for any vectorfield v^κ we have in ξ^κ: $\delta v^\kappa=dv^\kappa+S_{\mu\lambda}^{\cdot\cdot\kappa}v^\lambda d\xi^\mu$. Three special cases are considered. (1) $d\xi^\kappa$ is arbitrary and the λ-rank of $S_{\mu\lambda}^{\cdot\cdot\kappa}$ is $r\leqq n-1$. Then for $n>2$ the directions of the vectors v^κ for which $\delta v^\kappa=dv^\kappa$ in $\underset{0}{\xi}^\kappa$ for every choice of $d\xi^\kappa$ span an E_{n-r}. (2) The direction of v^κ is arbitrary and the λ-rank of $S_{\mu\lambda}^{\cdot\cdot\kappa}$ is $r\leqq n-1$. Then for $n>2$ the possible directions of $d\xi^\kappa$ for which $\delta v^\kappa=dv^\kappa$ in $\underset{0}{\xi}^\kappa$ for every choice of v^κ span an E_{n-r} in $\underset{0}{\xi}^\kappa$ and the E_{n-r}-field is X_{n-r}-forming. (3) $d\xi^\kappa$ and the direction of v^κ are fixed and the $\mu\lambda$-rank (equal to the κ-rank) of $S_{\mu\lambda}^{\cdot\cdot\kappa}$ is $r\leqq n-1$. For $n\geqq 4$ the bivectors $v^{[\lambda}d\xi^{\mu]}$ in $\underset{0}{\xi}^\kappa$ for which $S_{\mu\lambda}^{\cdot\cdot\kappa}v^\lambda d\xi^\mu=0$ form an $(\frac{1}{2}n(n-1)-r)$-parameter family. Some cases of special interest are considered also for half-symmetric connections $(S_{\mu\lambda}^{\cdot\cdot\kappa}=S_{[\mu}A_{\lambda]}^\kappa)$. *J. A. Schouten* (Epe).

Nožička, František. Le vecteur affinonormal et la connexion de l'hypersurface dans l'espace affin. Časopis Pěst. Mat. Fys. 75, 179–209 (1950). (French. Czech summary)

The author studies here the affine normals and affine connexions of a hypersurface in a space with symmetric affine connexion, in the light of the works of J. A. Schouten, D. J. Struik [Schouten, Der Ricci-Kalkül . . . , Springer, Berlin, 1924; Schouten and Struik, Einführung in die neueren Methoden der Differentialgeometrie, Noordhoff, Groningen-Batavia, v. I, 1935, v. II, 1938] and V. Hlavatý [Math. Z. **38**, 283–300 (1934); Nederl. Akad. Wetensch., Proc. **38**, 281–286, 738–743, 1006–1011 (1935)].

In §1, the author considers an n-dimensional space A_n with a symmetric affine connexion $\Gamma^\lambda_{\mu\nu}(\xi^\kappa)$, and a hypersurface X_{n-1}: $\xi^\lambda=\xi^\lambda(\eta^a)$ in A_n $(\lambda, \mu, \cdots=1, 2, \cdots, n;$ $a, b, \cdots=1, 2, \cdots, n-1)$. Supposing the rank of $B_a^\lambda=\partial_a\xi^\lambda$ to be $n-1$, he calls the vector t_λ satisfying $B_a^\lambda t_\lambda=0$, $t_\lambda\neq 0$ the tangent (covariant) vector of X_{n-1}. The general solution of these equations is given by ${}^*t_\lambda=Pt_\lambda$, $P\neq 0$. He calls each vector n^λ, defined on X_{n-1}, which satisfies $n^\lambda t_\lambda=1$ and whose law of transformation under ${}^*t_\lambda=Pt_\lambda$ is ${}^*n^\lambda=P^{-1}(n^\lambda+s^\lambda)$,

s^λ being a certain vector satisfying $s^\lambda t_\lambda = 0$, the affine normal (contravariant) vector of X_{n-1}. He then defines the tensor $h_{bc} = h_{cb} = B_b{}^\lambda \nabla_c t_\lambda$ and assumes that the rank of h_{bc} is $n-1$, where ∇_c denotes covariant differentiation along X_{n-1}. It is easily seen that the determinants $|B_a{}^\lambda, n^\lambda|$ and $|\nabla_a t_\lambda, t_\lambda|$ are both different from zero. Hence, when t_λ is fixed, if one defines the vector n^λ by $n^\lambda t_\lambda = 1$ and $n^\lambda \nabla_a t_\lambda = v_a$, v_a being a given covariant vector field in X_{n-1}, then n^λ is an affine normal, in the sense of the above definition.

In §2, the author first defines the quantities $B_\lambda{}^a$ by $B_\lambda{}^a B_b{}^\lambda = \delta_b^a$ and $B_\lambda{}^a n^\lambda = 0$. He then proves that

$$\Gamma^a{}_{bc} = \Gamma^a{}_{cb} = B_\lambda{}^a \nabla_b B_c{}^\lambda = h^{ad}(\nabla_d t_\lambda)\nabla_b B_c{}^\lambda + h_{bc} h^{ad} v_d \quad (v_d = n^\lambda \nabla_d t_\lambda)$$

are components of an affine connexion of X_{n-1} and calls it induced (by n^λ) affine connexion of X_{n-1}, where h^{bc} are defined by $h_{ab}h^{bc} = \delta_a^c$. From the Gauss equations $\Gamma^a{}_{bc} B_a{}^\lambda = h_{bc} n^\lambda + \nabla_b B_c{}^\lambda$, he obtains

$$n^\lambda = h^{bc}(B_a{}^\lambda \Gamma^a{}_{bc} - \nabla_b B_c{}^\lambda)/(n-1).$$

Conversely, if we define n^λ by the latter equation (v_d being regarded as given), then n^λ gives an affine normal. In particular, if one denotes by $\overset{0}{n}{}^\lambda$ the affine normal vector satisfying $\overset{0}{n}{}^\lambda t_\lambda = 1$, $\overset{0}{n}{}^\lambda \nabla_a t_\lambda = 0$ $(v_a = 0)$ and by $\overset{0}{\Gamma}{}^a{}_{bc}$ the components of the affine connexion induced by $\overset{0}{n}{}^\lambda$, then one has $\overset{0}{n}{}^\lambda = h^{bc}(B_a{}^\lambda \overset{0}{\Gamma}{}^a{}_{bc} - \nabla_b B_c{}^\lambda)/(n-1)$. The connexion $\overset{0}{\Gamma}{}^a{}_{bc}$ is called the inner connexion of X_{n-1}.

In §3, the author assumes that there is given an affine connexion $\Phi^a{}_{bc}$ on X_{n-1} and proves first that the vector defined by $N^\lambda = h^{bc}(B_a{}^\lambda \Phi^a{}_{bc} - \nabla_b B_c{}^\lambda)/(n-1)$ is an affine normal vector. This normal is said to be tied to the connexion $\Phi^a{}_{bc}$. It is also proved that the affine connexion $\overline{\Phi}{}^a{}_{bc}$ induced by N^λ is given by $\overline{\Phi}{}^a{}_{bc} = \overset{0}{\Gamma}{}^a{}_{bc} + h_{bc}h^{de}(\Phi^a{}_{de} - \overset{0}{\Gamma}{}^a{}_{de})/(n-1)$. The author also finds a necessary and sufficient condition that the given connexion $\Phi^a{}_{bc}$ be an induced one. If we induce an affine connexion to X_{n-1}, then a curve which is on X_{n-1} and is a geodesic of A_n is also a geodesic of X_{n-1}. In §4 the author considers the Riemannian connexion $\{{}^{\,a}_{bc}\}$ in X_{n-1} defined by h_{bc} and puts $m^\lambda = h^{bc}(B_a{}^\lambda \{{}^{\,a}_{bc}\} - \nabla_b B_c{}^\lambda)/(n-1)$, the connexion $I^a{}_{bc}$ induced by m^λ being given by

$$I^a{}_{bc} = \overset{0}{\Gamma}{}^a{}_{bc} + h_{bc}h^{de}(\{{}^{\,a}_{de}\} - \overset{0}{\Gamma}{}^a{}_{de})/(n-1).$$

He sets

$$T^a{}_{bc} = \{{}^{\,a}_{bc}\} - \overset{0}{\Gamma}{}^a{}_{bc},\quad M_b = 2T^c{}_{bc}/(n+1),\quad N_b = 2h_{bc}h^{de}T^c{}_{de}/(n+1),$$

and assumes that $R_b = 2B_b{}^\mu h^{ac} B_a{}^\lambda B_c{}^\nu R_{\lambda\mu\nu}{}^\omega t_\omega/(n+1) \neq 0$, $R_{\lambda\mu\nu}{}^\omega$ being curvature tensor of A_n. In general, the connexions $\overset{0}{\Gamma}{}^a{}_{bc}$ and $\{{}^{\,a}_{bc}\}$ do not coincide. Now, under the transformation ${}^*t_\lambda = Pt_\lambda$, the quantities $\overset{0}{\Gamma}{}^a{}_{bc}$, $\overset{0}{n}{}^\lambda$, $\{{}^{\,a}_{bc}\}$, m^λ, M_b, N_b are, respectively, transformed into

$$\begin{gathered}{}^*\overset{0}{\Gamma}{}^a{}_{bc} = \overset{0}{\Gamma}{}^a{}_{bc} - h_{bc}h^{ad}P_d,\quad {}^*\overset{0}{n}{}^\lambda = P^{-1}(\overset{0}{n}{}^\lambda - B_c{}^\lambda h^{cd}P_d),\\ {}^*\{{}^{\,a}_{bc}\} = \{{}^{\,a}_{bc}\} + \tfrac{1}{2}(\delta_b^a P_c + \delta_c^a P_b - h_{bc}h^{ad}P_d),\\ {}^*m^\lambda = P^{-1}(m^\lambda - [(n-3)/2(n-1)]B_c{}^\lambda h^{cd}P_d),\\ {}^*M_b = M_b + P_b,\quad {}^*N_b = N_b + P_b,\end{gathered}$$

where $P_a = \partial_a \log P$. Moreover, $M_b - N_b = R_b$, and, for $n=3$, the direction of m^λ is invariant.

In §5, the author proves first that a necessary and sufficient condition that n^λ be invariant under ${}^*t_\lambda = Pt_\lambda$ is that v_a have the transformation law ${}^*v_a = v_a + P_a$. Then, one can see that the connexions $\underset{0}{\Lambda}{}^a{}_{bc} = \overset{0}{\Gamma}{}^a{}_{bc} + h_{bc}h^{ad}M_d$ and $\underset{1}{\Lambda}{}^a{}_{bc} = \overset{0}{\Gamma}{}^a{}_{bc} + h_{bc}h^{ad}N_d$ are both invariant under ${}^*t_\lambda = Pt_\lambda$. The affine connexion $\underset{0}{\Lambda}{}^a{}_{bc}$ and the affine normal vector $\underset{0}{n}{}^\lambda$ tied to it are both said to be principal. The principal connexion $\underset{0}{\Lambda}{}^a{}_{bc}$ coincides with the one introduced by V. Hlavatý [loc. cit., p. 741]. In §6, the author discusses Weyl's connexion of X_{n-1} given by $\Psi^a{}_{bc} = \{{}^{\,a}_{bc}\} - \tfrac{1}{2}(\delta_b^a M_c + \delta_c^a M_b - h_{bc}h^{ad}M_d)$ which is invariant under ${}^*t_\lambda = Pt_\lambda$. In the last section, the author treats the case in which $R_{\lambda\mu\omega}{}^\alpha = 0$. In this case, M_b is a gradient, and consequently, if one sets $M_b = -\partial_b \log p$, then the transformation law of the vector $u_\lambda = pt_\lambda$ is ${}^*u_\lambda = cu_\lambda$, c being a constant. Thus, if one sets $g_{bc} = B_b{}^\lambda \nabla_c u_\lambda$,

$$\underset{u}{\overset{0}{\Gamma}}{}^a{}_{bc} = g^{ad}(\nabla_d u_\lambda)\nabla_b B_c{}^\lambda,\quad \underset{u}{n}{}^\lambda = g^{bc}(B_a{}^\lambda \overset{0}{\Gamma}{}^a{}_{bc} - \nabla_b B_c{}^\lambda)/(n-1),$$

then $\underset{u}{\overset{0}{\Gamma}}{}^a{}_{bc}$ and $\underset{u}{n}{}^\lambda$ are invariant under ${}^*t_\lambda = Pt_\lambda$ and $\underset{u}{n}{}^\lambda$ satisfies $\underset{u}{n}{}^\lambda t_\lambda = 1$, $\underset{u}{n}{}^\lambda \nabla_a u_\lambda = 0$. The affine normal $\underset{u}{n}{}^\lambda$ coincides with the one defined by J. A. Schouten [loc. cit., p. 145].

K. Yano (Princeton, N. J.).

Liber, A. E. On the classification of an affine connection in a space of two dimensions. Mat. Sbornik N.S. **27(69)**, 249–266 (1950). (Russian)

Soit X_2 l'espace courbe à deux dimensions, où soit ξ^λ $(\lambda = 1, 2)$ les coordonnées du point et $\Gamma^\nu_{\lambda\mu}$ $(\lambda, \mu, \nu = 1, 2)$ l'objet de la connexion affine de cet espace. Soit $E_2(\xi^\nu)$ l'espace tangent de l'espace X_2 en ξ^ν et enfin soient x^λ les coordonnées d'un point de $E_2(\xi^\nu)$. Le parallelisme défini par $dx^\kappa + \Gamma^\kappa_{\mu\lambda} x^\lambda d\xi^\mu = 0$, le long de la courbe $\xi^\nu = f^\nu(t)$, $t_1 < t < t_2$, dans l'espace X_2 implique une correspondance entre $E_2(\xi^\nu(t))$ et $E_2(\xi^\nu(t_0))$, $t_1 < t_0 < t_2$. Les équations de la correspondance sont les solutions du système des équations mentionné et elles ont la forme (1) $x^\lambda = P_\kappa{}^\lambda(t, t_0)\cdot x_0{}^\kappa$. Si nous utilisons la courbe fermée, qui passe par $\xi^\nu(t_0)$, nous obtiendrons en partant de (1) la transformation A de l'espace $E_2(\xi^\nu(t_0))$ en le même. L'ensemble de toutes les transformations A, qu'on peut obtenir en partant de toutes les courbes fermées qui passent par $\xi^\nu(t_0)$ est un groupe G, dit le groupe des holonomies en $\xi^\nu(t_0)$. On peut étendre ce groupe à l'aide de l'affinité centrale en groupe des holonomies dans tout l'espace X_2. Ce groupe est soit le groupe des affinités centrales soit un de ses sous-groupes. En utilisant la classification des sous-groupes du groupe des affinités et la relation entre les groupes continus de Lie et les objets géometriques dûe à V. V. Vagner [C. R. (Doklady) Acad. Sci. URSS (N.S.) **46**, 347–349 (1945); **53**, 183–186 (1946); ces Rev. **7**, 265; **8**, 404], on peut démontrer: Chaque sous-groupe du groupe des affinités centrales est déterminé par l'existence des quelques grandeurs caractéristiques dont les composantes ne changent pas pendant toutes les transformations du sous-groupe considéré.

À l'aide de l'identité de Ricci on trouve les conditions nécessaires et suffisantes pourque le groupe G appartienne au certain type. L'auteur a obtenu dix groupes des holonomies et les espaces X_2 correspondants (aussi l'espace de Berwald) et l'espace X_2 donné à la connexion affine avec la courbure nulle et pour lequel le groupe des holonomies est le groupe des identités.

Ensuite l'auteur étudie l'equivalence de la connexion affine et du groupe des mouvements. Le mouvement est la transformation de l'espace qui conserve l'objet de la connexion affine. La transformation infinitésimale du groupe des mouvements est déterminée par l'équation $\overset{L}{D}\Gamma^\kappa_{\mu\lambda} = 0$, ou $\overset{L}{D}$ est la dérivée de Lie. La méthode de Cartan donne la

possibilité de trouver deux comitantes différentielles d'objet de la connexion affine, qui sont les vecteurs linéairement indépendants. Les divers cas de la construction des comitantes considérées donnent la classification des objets de la connexion et parsuite la classification des espaces X_2. L'auteur distingue 4 cas, chaque d'eux a 5 types. (1) L'espace X_2 avec la connexion dont le courbure est égale à zero (l'affineur de Ricci est nul). On a le théorème: Chaque comitante différentielle de l'objet de la connexion dont la courbure est nulle est la fonction du vecteur de courbure et de ses dérivées covariantes jusqu'au troisième ordre. (2) L'espace X_2 avec la connexion quasieuclidienne. Dans ce cas l'affineur de Ricci n'est pas le bivecteur nul et on a $R_{(\lambda\mu)}=0$. (3) L'espace X_2 avec la connexion equivoluminaire. L'identité de Ricci a la forme $V_{\lambda\mu}=0$ et l'affineur de Ricci est le tenseur. (4) L'espace X_2 avec la connexion générale. La classification montre que chaque comitante différentielle d'objet de la connexion affine dans l'espace à deux dimensions est la fonction de l'affineur de Ricci, du vecteur de la courbure et des dérivées covariantes de ces grandeurs jusqu'au troisième ordre. *F. Vyčichlo* (Prague).

Levine, Jack. Collineations in generalized spaces. Proc. Amer. Math. Soc. **2**, 447–455 (1951).

This paper is a contribution to the study of a generalized space of paths H_n which is characterized by n functions $H^i(x, dx)$ which are homogeneous of the second degree in $dx^1, dx^2, \cdots, dx^n$, and whose paths are defined by

$$d^2x^i/ds^2+H^i(x, dx/ds)=0.$$

A collineation in an H_n is defined as a point transformation $\bar{x}^i=\bar{x}^i(x)$ which transforms paths into paths. The study of collineations in such spaces was initiated by Knebelman [Amer. J. Math. **51**, 527–564 (1929)]. The author obtains a simplified form of the collineation equations obtained by other authors, and applies this simplified form to a detailed study of collineations in the two-dimensional spaces H_2. He summarizes the results of the paper in a general theorem: If a (non-reducible) two-dimensional generalized space H_2 admits a real continuous group G_r of projective or affine collineations, then $r\leqq 3$. There are nine such complete groups of collineations, one G_1, two G_2, and six G_3.
E. T. Davies (Southampton).

Clark, R. S. The conformal geometry of a general differential metric space. Proc. London Math. Soc. (2) **53**, 294–309 (1951).

The object of the present paper is to give an account of the conformal theory in a general metric space based on the fundamental function $L(x, u)$, where u is a contra- or covariant relative vector of weight p or $-p$ [Schouten and Haantjes, Monatsh. Math. Phys. **43**, 161–176 (1936); E. T. Davies, Proc. London Math. Soc. (2) **49**, 241–259 (1947); these Rev. **8**, 491]. After a brief explanation of properties of the space, the author studies the effect of a conformal transformation $g'_{ij}=e^{2\sigma}g_{ij}$ (σ is an arbitrary function of position only) upon the fundamental objects of the space, and finds the quantity α_r which satisfies the equation $\alpha_r(x, u)-\alpha_r'(x, u)=\sigma_r(x)$. Using α_r to eliminate σ_r, he introduces the conformal derivative and differential defined for relative-conformal tensors. By a relative-conformal tensor we understand a tensor such that $T'^{(r)}_{(s)}=e^{h\sigma}T^{(r)}_{(s)}$ under a conformal transformation. A relative-conformal tensor is called conformetric, if $h=s-r+wn$ (s and r are numbers of contra- and covariant indices of the tensor, respectively, and w is its weight). The conformetric connection defined for conformetric tensors is then introduced, and it is shown that its curvature tensor is conformally invariant. Finally, using Lie derivatives, the conformal analogues of Killing's equations are stated and these give the conditions that the infinitesimal transformation is conformal.
A. Kawaguchi (Sapporo).

Brickell, F. On metrical geometries based on an integral as fundamental invariant. Proc. London Math. Soc. (2) **53**, 280–293 (1951).

The present paper deals with the metric geometry based on the double integral $\int\int L(x, u)dt^1dt^2$, where u^{ij} (or u_{ij}) is a simple contravariant (or covariant) bivector. This geometry has been discussed first by the reviewer [Proc. Imp. Acad. Tokyo **16**, 313–319, 320–325 (1940); these Rev. **2**, 167]. Supposing the bivector u^{ij} to be of weight p (or $-p$), the author derives first the metric bitensor $g_{ij,hk}$ from a fixed form of the function L. However, unfortunately, there is indeterminacy of the form of L because of the simplicity of u^{ij} and the author does not determine the metric bitensor because of this indeterminacy but makes only a remark concerning this problem which has been completely solved by the reviewer [Tensor (1) **6**, 49–61 (1943); N.S. **1**, 14–45 (1950); these Rev. **9**, 205; **12**, 536]. Then the author proves the theorems: (1) The equations $M_{rs}du^{rs}=0$ and $M_{rs}n^{rsi_3\cdots i_n}=0$ admit a unique solution $M_{rs}=0$, where $n^{rsi_3\cdots i_n}$ is the adjoint $(n-2)$-vector of the unit bivector $l^{ij}=u^{ij}/Lg^{p/2}$ and g^{n-1} is the $\frac{1}{2}n(n-1)$-rowed determinant of $g_{ij,hk}$ $(i<j, h<k)$ (these equations are equivalent to the equations $M_{rs}du^{rs}=0$ and $M_{[rs}u_{hk]}=0$; see the paper by the reviewer, loc. cit.); (2) Necessary and sufficient conditions for a bitensor $a_{ij,hk}$ to split up into the form $a_{ih}a_{jk}-a_{ik}a_{jh}$ are (a) $a_{ij[ab}a_{|hk|cd]}=0$,

(b) $$a_{ij[ab}a_{|hk|cd]}=a_{ih[ab}a_{|kj|cd]}=a_{ik[ab}a_{|jh|cd]}$$

with a, b, c, d arbitrary, and where the i, j, h, k are all different, and (c) for each set i, j, h, k there exists a set of values for a, b, c, d such that $a_{ij[ab}a_{|hk|cd]}\neq 0$. [Reviewer's remark: The phrase "the i, j, h, k are all different" in (b) of the last theorem must be replaced by "i, j, h, k are arbitrary" and therefore (a) is contained in (b). Because, if this is not so, the conditions are not invariant. In the proof of invariance of the conditions the author makes a mistake.] At the end of the paper there is a brief explanation of covariant differential, curvatures, and the theory of subspaces when the metric bitensor splits up, i.e., the space is of the metric class. A large part of the paper seems to be only a formal generalization of a part of R. Debever's results [Sur une classe d'espaces à connexion euclidienne, Thèse, Bruxelles, 1947; these Rev. **9**, 379] to an n-dimensional space, following an idea of E. T. Davies [J. London Math. Soc. **20**, 163–171 (1945); these Rev. **8**, 96]. *A. Kawaguchi*.

✓ ★**Finsler, P. Über Kurven und Flächen in allgemeinen Räumen.** Verlag Birkhäuser, Basel, 1951. x+160 pp. 12 Swiss francs; bound 14.80 Swiss francs.

Unveränderter Nachdruck of the author's Göttingen dissertation, 1918. A rather extensive bibliography through 1949 prepared by H. Schubert is appended.

Busemann, Herbert. On geodesic curvature in two-dimensional Finsler spaces. Ann. Mat. Pura Appl. (4) **31**, 281–295 (1950).

The main purpose of this paper is to introduce a geodesic curvature for curves in two-dimensional Finsler space, different from the classical one of Underhill and Landsberg. The

definition makes use of a new definition of angle in the local Minkowskian plane. The first variation of arc can be expressed in terms of this new geodesic curvature in a simple form. As a consequence the solutions of the isoperimetric problem, which are curves of given length bounding the maximum area, are to be found among curves whose geodesic curvature is a constant times the curvature of the local isoperimetrix at a point whose tangent is parallel to the tangent of the given curve. The author then discusses the possibility of defining Gaussian curvature in terms of this geodesic curvature and proves a weak form of the Gauss-Bonnet Theorem. Finally, discussions are given on the non-admissibility of normal coordinates in the general case. The author shows by examples that the difficulties cannot be removed by mere differentiability hypotheses. *S. Chern.*

Iwamoto, Hideyuki. On the relation between homological structure of Riemannian spaces and exact differential forms which are invariant under holonomy groups. I. Tôhoku Math. J. (2) **3**, 59–70 (1951).

Le groupe d'holonomie considéré ici est le groupe d'holonomie homogéne de la variété riemannienne V_m. Par suite les formes différentielles extérieures invariantes envisagées sont à dérivée covariante nulle; en particulier elles sont harmoniques. La plupart des resultats établis ici sont des conséquences simples de cette remarque et sont partiellement identiques à des résultats établis antérieurement par le rapporteur [C. R. Acad. Sci. Paris **230**, 1248–1250 (1950); **231**, 1413–1415 (1950); ces Rev. **11**, 741; **12**, 535]. Les deux cas principaux examinés par l'auteur sont celui où le groupe d'holonomie est réductible et laisse invariant un p-plan et celui où $m=2n$ et où le groupe laisse fixe une forme quadratique extérieure F de rang maximum. Le premier cas correspond aux variétés localement réductibles du rapporteur. Le second aux variétés localement kähleriennes. Dans le premier cas il est établi que $B_p(V_m)$ est non nul et dans le second que $B_{2k}(V_{2n})$ est non nul. On sait que ce dernier résultat est encore vrai si la forme F de rang maximum est seulement fermée. *A. Lichnerowicz* (Paris).

Guggenheimer, Heinrich. Quelques propriétés des variétés kählériennes closes. C. R. Acad. Sci. Paris **232**, 1398–1400 (1951).

The main result of this paper states that any pure closed form on a compact Kähler manifold is harmonic. The author has stated in a letter to the reviewer that the result is incorrect. *W. V. D. Hodge* (Cambridge, England).

Hodge, W. V. D. Differential forms on a Kähler manifold. Proc. Cambridge Philos. Soc. **47**, 504–517 (1951).

In complex structure consider a differential form

$$P=\sum_{(\alpha),(\beta)}A_{\alpha_1\cdots\alpha_r\beta_1^*\cdots\beta_s^*}dz_{\alpha_1}\cdots dz_{\alpha_r}d\bar{z}_{\beta_1}\cdots d\bar{z}_{\beta_s}$$

for a given type (r, s) and denote by dP its external derivative in the z_α holding the $\bar{z}_\beta$ fixed and by $\bar{d}P$ its derivative in the $\bar{z}_\beta$ holding the z_α fixed. The most quotable results of the paper are as follows. On a compact Kähler manifold, if $dP=0$ then $P=h+d\varphi$, where h is a harmonic form of type (r, s) and φ is of the type $(r-1, s)$. Also, it is possible to make $h=0$ if and only if $(P, h')=0$ for all harmonic h' of type (r, s). Furthermore, if both $dP=0$ and $d\bar{P}=0$, then $P=h+d\bar{d}A$ where A is of type $(r-1, s-1)$, and again $h'=0$ if and only if $(P, h')=0$.
S. Bochner (Princeton, N. J.).

Bochner, S. Complex spaces with transitive commutative groups of transformations. Proc. Nat. Acad. Sci. U. S. A. **37**, 356–359 (1951).

The author proves the following theorem. If a complex coordinate space V_{2k} admits r $(r\geqq k)$ holomorphic contravariant vector fields η_p^α, $\alpha=1, \cdots, k$, $p=1, \cdots, r$ with $\|\eta_p^\alpha\|$ of rank k, and if $\eta_p^\beta\partial\eta_q^\alpha/\partial z^\beta-\eta_q^\beta\partial\eta_p^\alpha/\partial z^\beta$ vanish over V_{2k}, then there exist on V_{2k} k simple abelian integrals of the first kind by which it is mapped holomorphically and locally one-to-one into euclidean E_{2k}; in particular, if V_{2k} is compact it is a complex multi-torus. The proof is accomplished by introducing a flat Kähler metric on V_{2k}.
S. B. Myers (Ann Arbor, Mich.).

Libermann, Paulette. Sur la courbure et la torsion des variétés presque hermitiennes. C. R. Acad. Sci. Paris **233**, 17–19 (1951).

[This is a continuation of another note by C. Ehresmann and the author [same C. R. **232**, 1281–1283 (1951); these Rev. **12**, 749].] An almost hermitian structure in a manifold V^{2n} is defined by n independent complex-valued differential forms ω_i, with $ds^2=\sum\omega_i\bar{\omega}_i$. From the curvature and torsion forms, one constructs conformal curvature and first torsion forms which, under multiplication of the metric by a scalar function, are invariant, resp. also multiplied by a scalar. Vanishing of these two are necessary and (for $n>2$) sufficient for ds^2 to be expressible as $\rho(z_1, \cdots, z_n)(dz_1d\bar{z}_1+\cdots+dz_nd\bar{z}_n)$. The scalar curvature for a (real) plane element at a point is defined similar to the Riemannian case. Theorem I: If at each point the scalar curvature is independent of the plane element, then it has the value zero. Theorem II: The almost Kähler structures with vanishing scalar curvature are locally equivalent to linear space. A scalar torsion for a (complex) plane element at a point is defined. Theorem III: If the first or second torsion vanishes, and the scalar curvature and trosion at each point are independent of the plane element, then the structure is locally equivalent either to linear space, or to a certain structure on the (complex) orthogonal group in three parameters.
H. Samelson (Ann Arbor, Mich.).

Eckmann, Beno, et Frölicher, Alfred. Sur l'intégrabilité des structures presque complexes. C. R. Acad. Sci. Paris **232**, 2284–2286 (1951).

An almost complex structure in a manifold V^{2n} can be given by a tensor a_k^j with $a_k^ia_i^h=-\delta_i^j$. The structure is integrable if it is induced by a complex coordinate system; the analytic structure is then completely determined. With $a^j_{kl}=\partial a_k^j/\partial x^l-\partial a_l^j/\partial x^k$ one defines $t^j_{kl}=a^j_{pk}a_l^p-a^j_{pl}a_k^p$; this can be shown to be a tensor. Theorems 1 and 2 state that vanishing of this tensor is necessary and (assuming class C^ω) sufficient for integrability. Necessity is obvious. Sufficiency is shown by setting up a system of total differential equations, which are integrable by virtue of $t^j_{kl}=0$, and whose solutions can be used to construct the required complex coordinates. A certain almost complex structure on S^6 is shown to be not integrable, since its t^j_{kl} does not vanish. [Related results have been established by C. Ehresmann and P. Libermann [C. R. Acad. Sci. Paris **232**, 1281–1283 (1951); these Rev. **12**, 749], C. Ehresmann, G. deRham (unpublished), E. Calabi and D. C. Spencer [Bull. Amer. Math. Soc. **57**, 254 (1951)].]
H. Samelson (Ann Arbor, Mich.).

Morinaga, Kakutarô, and Nôno, Takayuki. On the automorphisms of the set of simple vectors. J. Sci. Hiroshima Univ. Ser. A. 15, 11–24 (1951).

It is proved that any automorphism of the set of all simple r-vectors in R_n,

$$'w_{i_1\cdots i_r}=p^{j_1\cdots j_r}_{i_1\cdots i_r}w_{j_1\cdots j_r},$$

can for $r\neq\frac{1}{2}n$ be reduced to an automorphism of vectors in R_n, $'w_i=v_i^jw_i$, and for $r=\frac{1}{2}n$ either to an automorphism of vectors or to such an automorphism followed by a transformation carrying each r-vector into an r-vector of the same size perpendicular to the first:

$$'w_{i_1\cdots i_r}=g_{i_1l_1}\cdots g_{i_rl_r}e^{k_1\cdots k_rl_1\cdots l_r}v^{j_1}_{k_1}\cdots v^{j_r}_{k_r}w_{j_1\cdots j_r}.$$

The proof depends on a classification of automorphisms.

J. A. Schouten (Epe).

Finkelstein, R., LeLevier, R., and Ruderman, M. Nonlinear spinor fields. Physical Rev. (2) 83, 326–332 (1951).

The author examines some solutions of a set of nonlinear spinor differential equations derived from a Lagrangean. The spinor field is assumed to have a simple angular and temporal dependence and satisfy the boundary conditions: the fields are regular and all observable integrals are finite. These boundary conditions lead to a nonlinear proper value problem whose solutions can be discussed in the phase plane. Numerical solutions for the spinor field were obtained with a differential analyzer. Properties of various observables computed in terms of these solutions are discussed.

A. H. Taub (Urbana, Ill.).

Gardner, G. H. F. Canonical coordinates at a point for two skew-symmetric tensors. Proc. Amer. Math. Soc. 2, 328–334 (1951).

This paper deals with a pair of skew-symmetric tensors, $P_{mn}=-P_{nm}$, $Q_{mn}=-Q_{nm}$, in a space of four dimensions. The question is whether it is possible, by a nonsingular transformation with positive jacobian, to pass to a new coordinate system such that at a point A the following relations hold: (*) $P_{12}=Q_{34}$, $P_{23}=Q_{14}$, $P_{31}=Q_{24}$, $P_{14}=-Q_{23}$, $P_{24}=-Q_{31}$, $P_{34}=-Q_{12}$. Such a coordinate system is called P-canonical. Since only one point of space is involved, the problem is purely algebraic; it is closely connected with the development of nonmetrical electromagnetic theory [J. L. Synge, Proc. Symposia Appl. Math., vol. 2, pp. 21–48 (1950); these Rev. 11, 401]. The author introduces the following notation: $(\epsilon PQ)=\frac{1}{2}\epsilon^{mntu}P_{mn}Q_{tu}$, where ϵ^{mntu} is the usual permutation symbol, and η^{rs} = the diagonal array $(1, 1, 1, -1)$. Then the relations (*) may be written $\eta^{rm}\eta^{sn}P_{rs}=\frac{1}{2}\epsilon^{mntu}Q_{tu}$. [The factor $\frac{1}{2}$ is consistently omitted throughout the paper, without vitiating the results.] He shows that $(\epsilon PP)+(\epsilon QQ)=0$ is a necessary and sufficient condition for the existence of P-canonical coordinates, this condition being of course invariant under coordinate transformations. The result is obtained through two preliminary theorems which establish (a) that if P_{mn} and Q_{mn} are each of rank 2, it is possible to make all components vanish except (i) one in P_{mn} and one in Q_{mn} if $(\epsilon PQ)\neq 0$, (ii) two in P_{mn} and two in Q_{mn} if $(\epsilon PQ)=0$, and (b) that P-canonical coordinates exist if P_{mn} and Q_{mn} are each of rank 2. In all cases the tensors are assumed to be linearly independent.

J. L. Synge (Dublin).

Ishizuka, Ioshio. Rheonomic abstract geometry and basic equation of rotating electric machinery. Mem. Fac. Sci. Eng. Waseda Univ. 14, 51–52 (1950).

Whenever a rotating electrical machine is connected to a transmission system or to another rotating machine, it is necessary to introduce reference frames that are not tied rigidly to the conductors but follow the motion of some arbitrarily moving flux or voltage wave. As the dynamical system is thereby subjected to nonholonomic moving constraints, the question arises whether the angular displacement of the freely rotating frame should be considered as an additional generalized coordinate or as a parameter. The author assumes the parametric point of view and employs the "rheonomic geometry" of Wundheiler to show that the covariant derivative includes not only the affine connection of a non-Riemannian space but also a geometric object of valence two.

G. Kron (Schenectady, N. Y.).

NUMERICAL AND GRAPHICAL METHODS

✱Sluckiĭ, E. E. Tablicy dlya vyčisleniya nepolnoĭ Γ-funkcii i funkcii veroyatnosti χ^2. [Tables for the Computation of the Incomplete Γ-Function and the Probability Function χ^2]. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1950. 71 pp.

Let $T(\chi^2, n)=(\frac{1}{2}\chi^2)^{-\frac{1}{2}n}[1-P(\chi^2, n)]$, $\mathcal{P}(t, n)=P(\chi^2, n)$ and $\Pi(t, x)=P(\chi^2, n)$, where $t=(2\chi^2)^{\frac{1}{2}}-(2n)^{\frac{1}{2}}$, $\chi=(\frac{1}{2}n)^{-\frac{1}{2}}$, and

$$P(\chi^2, n)=\frac{1}{2^{\frac{1}{2}(n-2)}\Gamma(\frac{1}{2}n)}\int_\chi^\infty x^{n-1}e^{-\frac{1}{2}x^2}dx.$$

Table I lists $T(\chi^2, n)$ for

$$\chi^2=0(0.05)2(0.1)1;\quad n=0(0.05)2(0.1)6.$$

Table II lists $P(\chi^2, n)$ for $\chi^2=0(0.1)3.2$, $n=0(0.05)2(0.1)6$ and $\chi^2=3.2(0.2)7(0.5)10(1)35$, $n=0(0.1)0.4(0.2)6$. Table III lists $\mathcal{P}(t, n)$ for $t=-4(0.1)4.8$, $n=6(0.5)11(1)32$. Table IV lists $\Pi(t, x)$ for $t=-4.5(0.1)4.8$, $x=0(0.02)0.22(0.01)0.25$. Second and fourth central differences are given and entries are to 5 decimal places. Table V lists the Everett and the Newton interpolation coefficients.

✱Tables of the Bessel Functions of the First Kind of Orders Seventy-Nine Through One Hundred Thirty-Five, by the Staff of the Computation Laboratory. The Annals of the Computation Laboratory of Harvard University, vol. XIV. Harvard University Press, Cambridge, Mass., 1951. x+614 pp. $8.00.

This is the last volume of the monumental Harvard Tables of Bessel functions of the first kind of integral order. It contains 10 decimal tables of $J_n(x)$ for $n=79(1)135$ and $x=0(.01)99.99$, 18 decimal values of $J_n(100)$ for $n=0, 1, 2, 3$, a 10 decimal table of $J_n(100)$ for $n=0(1)135$, and a 15 decimal table of $J_n(n)$ for $n=0(1)100$.

In the preface, H. H. Aiken points out that although originally tabulation was envisaged up to $n=100$ only, in the course of the work it was found useful to extend the tables to $n=135$; for larger n, the Bessel function of the first kind vanishes to 10 decimal accuracy in the tabulated range of x. The opinion of the staff of the Computation Laboratory of Harvard University regarding the importance of this project is amply born out by the fact (mentioned in the preface) that similar tabulation was undertaken in Germany and Russia, and, one may add, also by the great interest

shown by all computers in the progress of the work. Aiken also remarks that the project has served to train computers, and that in its course improvements were made in machine design, computing techniques, and the process of producing numerical tables by automatic machinery.

The entire project staff must be congratulated on the completion of this set of tables. They not only fulfilled their original plan but even enlarged it as they went along, and produced a work which will remain, for a long time to come, an indispensable tool for computers. *A. Erdélyi.*

Luke, Yudell L., and Dengler, Max A. Tables of the Theodorsen circulation function for generalized motion. J. Aeronaut. Sci. **18**, 478–483 (1951).

The authors tabulate the function

$$C(z)=\frac{H_1^{(2)}(z)}{H_1^{(2)}(z)+iH_0^{(2)}(z)}=F(\rho,\theta)+iG(\rho,\theta),$$

where $z=\rho e^{i\theta}$, in the range

$$\rho:\ (0.01)0.3(0.02)0.5(0.05)1.0(0.5)10.0,$$

θ: $-5°(5°)30°$. This function is of importance in the theory of nonsteady airfoil theory. It is further stated that the present work corrects some earlier results on the same function given by W. P. Jones [Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2117** (1945); these Rev. **10**, 163]. *E. Reissner.*

***Kantorovič, L. V., i Krylov, V. I. Približennye metody vysšego analiza. [Approximate Methods of Higher Analysis].** 3d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 695 pp.

Only minor revisions and corrections of the 2nd 1941 edition have been made. Table of contents: (I) Methods based on representation of the solution as an infinite series; (II) Approximate solution of the Fredholm integral equation; (III) The method of nets; (IV) Variational methods; (V) Conformal mapping of regions; (VI) Principles of application of conformal mapping to the solution of basic problems for canonical regions; (VII) The method of Schwarz.

Kudryavcev, L. D. On the principles of carrying out arithmetical operations on computing machines. Uspehi Matem. Nauk (N.S.) **5**, no. 3(37), 104–127 (1950). (Russian)

This is an exposition of some elementary principles for designing devices for carrying out simple arithmetic operations on digital machines. The author discusses systems of numeration, machine elements (relays, vacuum tubes, etc.), registers, adders, multipliers, and dividers, with some very brief remarks about control. *H. B. Curry.*

Romberg, Werner. Approximation eines Kurvenstückes durch wenige sin-Funktionen. Avh. Norske Vid. Akad. Oslo. I. **1949**, no. 3, 10 pp. (1949).

The paper is concerned with describing an algorithm for finding an approximation $\sum_{k=1}^{N} b_k \sin s\rho_k$ to an experimentally determined function known to be of the form $\sum_{k=1}^{N} a_k \sin sr_k$ with $a_k>0$. The method is by successive approximations based on a least square technique in which one successively obtains improved values both for b_k and ρ_k. The procedure is described in some detail but no proof of the convergence is given. *H. Goldstine* (Princeton, N. J.).

Turán, Paul. On approximative solution of algebraic equations. Publ. Math. Debrecen **2**, 26–42 (1951).

Starting with the method of Dandelin-Lobatschevskij-Graeffe for the solution of an algebraic equation with numerical coefficients, the author develops rules for the computation of bounds for the moduli of the roots. A formula is also developed for the determination of the absolute value of the imaginary part of the root which has the imaginary part of greatest absolute value. *E. Frank.*

Gorodskiĭ, D. A. A simple method of numerical solution of algebraic equations. Èlektričestvo **1951**, no. 3, 65–66 (1951). (Russian)

A rough approximation to a pair of complex roots of the polynomial $P(z)$ is found by using a pair of ordinary draughting triangles to construct the points $P(z)$ for several constant values of $|z|$ and $\arg z=0°, 30°, 45°, \cdots, 180°$ (i.e. several points on curves that would be drawn by an isograph are plotted by vector addition). The corresponding quadratic factor z^2+az+b and also $z^2+(a+\Delta a)z+b$ and $z^2+az+(b+\Delta b)$, with Δa and $\Delta b=.01$ for example, are then divided into $P(z)$ to obtain approximate values of the partial derivatives of the remainder coefficients with respect to a and b. These are used to set up a pair of linear equations whose solution gives corrections for improving a and b. Convergence of the process is not discussed. Application to a biquadratic is shown. *R. Church.*

Sesini, Ottorino. Sull'approssimazione dei procedimenti energetici per il calcolo di autovalori. Aerotecnica **31**, 94–99 (1951).

17, (1950), 138–145 (1951)

Fichera, Gaetano. Sulla maggiorazione dell'errore di approssimazione nei procedimenti di integrazione numerica delle equazioni della fisica matematica. Rend. Accad. Sci. Fis. Mat. Napoli (4) ~~17, 8~~ pp. = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **289** (1950).

The purpose of the present note is to provide a practical method for estimating the error in the approximations made in solving boundary value problems by means of certain numerical procedures used by the "Istituto Nazionale per le Applicazioni del Calcolo". For the special case of the Dirichlet problem for a bounded three-dimensional closed domain D with a smooth boundary FD, the following two theorems are proved: (I) If D possesses a Green's function $G(P, Q)$, then there exists a positive number K such that

$$\int_D u^2 dxdydz \leqq K\cdot\int_{FD} u^2 d\sigma,$$

whenever the real-valued function u is continuous on D and harmonic on $D-FD$. (II) If D possesses a Neumann's function $N(P, Q)$, then there exists a positive number q such that

$$\int_{FD} u^2 d\sigma \leqq q^{-1}\int_D |\operatorname{grad} u|^2 dxdydz,$$

whenever the function u is continuous on D, harmonic on $D-FD$, and $\int_{FD} u d\sigma=0$. The constants K and q may be approximated numerically by employing a certain sequence of particular harmonic functions introduced earlier [Giorn. Mat. Battaglini (4) **2**(**78**), 71–80 (1948); these Rev. **10**, 606]. It is indicated how these results may be extended to other boundary value problems, in particular to those of three-dimensional elasticity. *J. B. Diaz* (College Park, Md.).

Ingram, W. H. A modification of Southwell's method. Quart. Appl. Math. 9, 314–315 (1951).

Jánossy, L. Search for periodicities. Acta Phys. Acad. Sci. Hungaricae 1, 36–55 (1951). (English. Russian summary)

To make a formal trigonometric analysis of a finite sequence of data, the author proposes a process of successive summation, in analogy to W. Schmidt's successive integration. The mean value of each of the sequences obtained by summation is taken equal to zero by appropriate choice of the constant of integration. The resulting formulas are simpler than Schmidt's, and the work is shorter than that in J. Fuhrich's autocorrelation method; however, some of this advantage is lost in applications to many of the sequences that occur in nature, in which secular trends and autocorrelations appear to restore some of the complexities which the author has eliminated. *A. Blake.*

***Pentkovskiĭ, M. V. Nomografiya. [Nomography].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949. 280 pp. (3 plates)

This book is intended to enable the engineer to construct nomograms with reasonable ease so that they will attain desired standards of accuracy. In the First Part (124 pp.) alignment diagrams are classified according to nomographic order and genus and their general theory is presented in descriptive terms with particular detail in the case of the third nomographic order. Coincidence nomograms, binary fields and binary scales then lead naturally to composite nomograms. The reader is referred to N. A. Glagolev [Theoretical Foundations of Nomography, 2d ed., GTTI, Moscow-Leningrad, 1936] for proofs. The author's concept of the characteristic of a scale [Doklady Akad. Nauk SSSR (N.S.) 66, 339–342 (1949); these Rev. 11, 406] is introduced and here extended to curve nets. Its relation to accuracy in general and its use in obtaining a so-called best projective transformation in the case of rectilinear scales is explained. The Second Part (116 pp.) of the book is devoted to practical methods for constructing nomograms. Here the principal tool is the author's method [Učenye Zapiski Moskov. Gos. Univ. Nomografiya 28, 115–140 (1939); these Rev. 1, 254] for studying projective transformations with various invariant elements by means of so-called networks. This method is explained in detail for certain nomograms of order 3 and 4. Network forms with invariant parallel lines and with an invariant triangle are provided as separate plates for application of the method. For convenient construction of a nomogram of genus two for the third order relation when it is in the canonical form $f_1+f_2=f_3$ and for transforming it to obtain maximum accuracy a "frame" consisting of an ellipse with major diameter is provided and its use explained. At the end of Part One a number of special topics are sketched, such as nomograms for systems of equations, for empirical data, for high accuracy, with repeated variables, etc. In the conclusion (21 pp.) some applications of d'Ocagne's theory of superimposed planes and contact are given [Traité de nomographie . . . , Gauthier-Villars, Paris, 1921, pp. 365–444].

Various sections of the book are quite independent of each other. No serious errors were noted. The material is well chosen for the purpose in view; examples are numerous, fully worked out and arranged to permit comparison. This systematic exposition of the author's methods against a background of standard material should make the book of value at least on the practical side of the subject. *R. Church.*

Liebmann, G. A method for the mapping of vector potential distributions in axially symmetrical systems. Philos. Mag. (7) 41, 1143–1151 (1950).

Der Verfasser baut die von ihm in früheren Abhandlungen [Nature 164, 149–150 (1949); British J. Appl. Phys. 1, 92–103 (1950)] entwickelte Methode von elektrischen Widerständen dahingehend aus, um sie auch für rotationssymmetrische Magnetfelder mit stromführenden Leitern und Eisenkernen verwenden zu können. *P. Funk.*

de Graaff-Hunter, J. The geodetic uses of gravity measurements and their appropriate reduction. Proc. Roy. Soc. London. Ser. A. 206, 1–17 (1951).

The author stresses the need of a uniform international reference system for the reduction of the geodetic measurements made by national organizations which are now reduced in a disconnected and arbitrary manner. A plea is made for the use of a world-wide net of gravity measurements fitted into the Stokes theorem [G. G. Stokes, Trans. Cambridge Philos. Soc. 8, 672–695 (1849)]. The author gives a method for expressing deflections of the vertical and curvature of the geoid by suitable differentiation of the Stokes formula and proposes astronomical observations on short base lines to determine the local deflection of the vertical and curvature. The reviewer would suggest caution in the employment of astronomical observations for this purpose unless checked in each case for local gravity anomalies by a detailed gravity-meter and torsion balance profile. The original article should be consulted for the mathematical development. *J. B. Macelwane* (St. Louis, Mo.).

RELATIVITY

√ ***Rainich, G. Y. Mathematics of Relativity.** John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1950. vii+173 pp. $3.50.

Cet ouvrage didactique constitue un exposé très clair de l'axiomatique de la relativité. Il ne vise pas à être complet, mais à donner des fondements de la théorie une image cohérente, et ce but est très sensiblement atteint. Après une introduction consacrée à la physique classique et initiant déjà à la notation tensorielle, l'auteur étudie en détail la géométrie de l'espace de Minkowski, c'est-à-dire celle de l'espace pseudo-euclidien de type hyperbolique normal à quatre dimensions. À ce chapitre mathématique succède le chapitre physique qui en est la traduction naturelle, celui consacré à la relativité restreinte. Le rapporteur note en particulier l'excellent paragraphe relatif à l'étude du "tenseur complet" T_{ij} et à la détermination à partir de ce tenseur des éléments matériels et électromagnétiques de la distribution d'énergie correspondante; cette préoccupation est rare dans les ouvrages didactiques concernant la relativité et est en connexion étroite avec des résultats classiques de Synge et du rapporteur. Selon le même plan général, les deux derniers chapitres sont consacrés l'un à la géométrie des espaces de Riemann, l'autre à la théorie relativiste de la gravitation. Le dernier chapitre comporte la formation du ds^2 de Schwarzschild et ses applications à la détermination d'effets relativistes observables: variation du périhélie de Mercure, courbure des rayons lumineux, déviation des raies spectrales. Une grande rigueur et un grand souci du détail font le prix

de ce livre dont des exercices illustrent heureusement les chapitres mathématiques. Il est seulement permis de regretter l'absence d'une bibliographie qui, sans avoir la prétention d'être complète, aurait jalonné et éclairé les principaux pas de la théorie. Le rapporteur aurait aussi aimé que quelque indication soit donnée sur le caractère global essentiel de la détermination des métriques einsteiniennes; les conditions locales données par le système aux dérivées partielles d'Einstein ne suffisent pas, par elles-mêmes, à déterminer les solutions; il convient d'y adjoindre des conditions de régularité globales, en particulier de raccordement des solutions intérieures et extérieures, et éventuellement la notion de comportement asymptotique euclidien. Cela est particulièrement net en ce qui concerne la formation du ds^2 de Schwarzschild, pour laquelle l'auteur semble un peu gêné dans son exposé. Les postulats adoptés ne sont pas tout-à-fait heureux comme le montre une étude globale du rapporteur [C. R. Acad. Sci. Paris **222**, 432–434 (1946); **226**, 775–777, 2119–2120 (1948); ces Rev. **7**, 397; **9**, 538; **10**, 157].

A. Lichnerowicz (Paris).

Jordan, Pascual. Vierdimensionale Begründung der erweiterten Gravitations-Theorie. Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. **1950**, 319–334 (1950).

This paper is concerned with a generalization of general relativity in which the gravitational constant is replaced by a scalar field K. The field equations are derived from the variational principle

$$\delta\int K^{\eta}\left(G-\xi\frac{K_{,k}K_{,l}g^{kl}}{K^2}\right)\sqrt{-g}\,d\tau=0,$$

where $K_{,k}=\partial K/\partial x^k$, η and ξ are constants and G is the scalar curvature of the Riemannian space with metric tensor g_{ij}. A spherically symmetric static solution of the field equations is given. In this case it is shown that K must vary slowly if at all if the precession of the perihelion of Mercury and the deflection of light by the sun are to be accounted for correctly. From considerations involving the gravitational field outside a spherical mass it is concluded that the constant ξ must be such that its absolute value is much greater than one. The author argues on the basis of assumptions concerning a simple cosmological model that $\eta=1$. The paper closes with an application of this theory to the statistical behavior of a collection of particles of various masses.

A. H. Taub (Urbana, Ill.).

Fourès-Bruhat, Yvonne. Théorèmes d'existence et d'unicité pour les équations de la théorie unitaire de Jordan-Thiry. C. R. Acad. Sci. Paris **232**, 1800–1802 (1951).

The author considers the field equations $R_{\alpha\beta}-\frac{1}{2}\gamma_{\alpha\beta}R=0$ $(\alpha,\beta=0,1,2,3,4)$, where $R_{\alpha\beta}$ is the Ricci tensor of a five-dimensional space with the metric

$$d\sigma^2=\gamma_{\alpha\beta}dx^\alpha dx^\beta=(Vdx^0+\beta\varphi_i dx^i)^2+g_{ij}dx^i dx^j\quad(i,j=1,2,3,4),$$

and V, φ_i and g_{ij} are independent of x^0. She states that these equations when looked upon as a Cauchy problem in partial differential equations have a unique solution for initial data of a certain type. It is also stated that if V=constant and $\partial V/\partial x^\nu=0$ for $x^\nu=0$ then V is constant throughout as a consequence of the field equations. *A. H. Taub.*

Tonnelat, M. A. Théorie unitaire affine du champ physique. J. Phys. Radium (8) **12**, 81–88 (1951).

This paper gives a connected account of results on a unified field theory using a nonsymmetrical affine connection. These results have been previously announced in a series of notes [C. R. Acad. Sci. Paris **228**, 368–370, 660–66, 1846–1848 (1949); **230**, 182–184 (1950); **231**, 470–472, 487–489, 512–514 (1950); these Rev. **10**, 408, 498; **11**, 146, 569; **12**, 291, 292]. It is shown that the Euler equations of a "hermitian" Lagrangian may be solved explicitly for the coefficients of the connection. The field equations due to a particular choice of the Lagrangian function are obtained and an approximate form of these equations is derived. The approximation consists in neglecting third and higher powers of the antisymmetric tensor describing the electromagnetic field.

A. H. Taub (Urbana, Ill.).

Wyman, Max. Unified field theory. Canadian J. Math. **2**, 427–439 (1950).

Ce papier est consacré à l'étude rigoureuse des solutions statiques à symétrie sphérique des équations du champ unitaire fournies par la théorie d'Einstein et Straus [Ann. of Math. (2) **47**, 731–741 (1946); ces Rev. **8**, 412]. Une telle étude avait été commencée par Papapetrou [Proc. Roy. Irish Acad. Sect. A. **52**, 69–86 (1948); ces Rev. **10**, 580] qui supposait même les équations du champ douées d'une constante cosmologique. Dans le présent papier, ce sont les équations sans constante cosmologique qui sont seules envisagées. La forme générale de tenseur $g_{\lambda\mu}$ cherché étant, en coordonnées polaires sphériques,

$$g_{\lambda\mu}=\begin{bmatrix}-\alpha & 0 & 0 & w\\ 0 & -\beta & r^2v\sin\theta & 0\\ 0 & -r^2v\sin\theta & -\beta\sin^2\theta & 0\\ -w & 0 & 0 & \gamma\end{bmatrix},$$

où α,β,γ,v,w sont des fonctions de r, Papapetrou a obtenu la solution générale des équations du champ dans le cas $v=0$, $w\neq0$. Dans une première partie l'auteur obtient la solution générale par $v\neq0$, $w=0$. Dans une seconde partie, il se préoccupe de l'interprétation physique des solutions obtenues et de la comparaison des résultats avec ceux de la théorie de la relativité générale. Le problème principal semble celui de la construction de la métrique d'espace-temps $a_{\lambda\mu}$ à partir des grandeurs de champ $g_{\lambda\mu}$ et $\Gamma^{\nu}{}_{\lambda\mu}$. L'auteur montre que l'indentification $a_{\lambda\mu}=g_{(\lambda\mu)}$ n'est nullement nécessaire, discute les différentes possibilités et montre même, par un exemple assez artificiel, qu'il est possible de choisir la métrique de façon à faire correspondre à l'une des solutions obtenues le ds^2 de Schwarzschild. Dans une dernière partie, l'auteur discute les conditions aux limites qu'il convient d'imposer aux solutions trouvées; certaines difficultés conduisent à penser que le problème physique de la particule chargée ne pourrait être résolu que par la connaissance des solutions relatives au cas plus général $vw\neq0$.

A. Lichnerowicz (Paris).

Lampariello, Giovanni. Relatività ed elettrodinamica. Boll. Un. Mat. Ital. (3) **6**, 118–142 (1951).

Dopo aver fatto cenno delle idee principali che caratterizzano lo sviluppo storico-critico dell'elettrodinamica in relazione al principio di relatività, l'autore rivolge l'attenzione al problema generale delle forze ponderomotrici nel campo elettromagnetico, di cui non si possiede ancora la soluzione.

Author's summary.

MECHANICS

Grüss, Gerhard. Zur Kinematik des Rollgleitens. Z. Angew. Math. Mech. **31**, 97–103 (1951). (German. English, French, and Russian summaries)

Die ebene Kurve C_1 bewegt sich längs der raumfesten Kurve C so dass die Kurven einander berühren, während für die Bogenlängen s_1 und s gilt $s_1=\lambda s$, wo λ eine Konstante ist. Die Bewegung wird als Rollgleiten bezeichnet ($\lambda=0$ ist der Fall des reinen Gleitens, $\lambda=1$ der des reinen Rollens). Verf. bestimmt die Polkurven der Bewegung. Sind $x=x(s)$, $y=y(s)$ die Gleichungen von C, $k(s)$ und $k_1(s)$ die Krümmung von C, bzw. C_1, x_P und y_P die Koordinaten des Pols P in die raumfeste Ebene, so hat man

$$x_P=x(s)-\frac{y'(s)(1-\lambda)}{k-\lambda k_1},\quad y_P=y(s)+\frac{k'(s)(1-\lambda)}{k-\lambda k_1}.$$

[Bemerkung des Referenten. Verf. erreicht sein Ergebnis nach einer ziemlich langen Rechnung und nennt das Resultat "erstaunlich einfach." Man kann statt seinen Gleichungen (4) die Tatsache benutzen dass P auf die gemeinsame Normale von C und C_1 im Berührungspunkt A liegt. Sind M and M_1 die Krümmungsmittelpunkte, R and R_1 die Krümmungsradien, so ist die Geschwindigkeit von M_1 gleich $v_1=(R-R_1)\,\dot{s}/R$ und diejenige von A gleich $v_A=\dot{s}-\dot{s}_1$. Das Resultat findet man dann fast ohne Rechnung aus der Beziehung $AP:M_1P=v_A:v_1$. Auch braucht λ nicht konstant zu sein; des Ergebnis ist auch gültig für veränderliche λ wenn man λ durch $\dot{s}_1/\dot{s}$ erklärt.] *O. Bottema* (Delft).

Zanaboni, Osvaldo. Dimostrazione e discussione della legge delle rigidezze. Ann. Triestini. Sez. 2 (4) **3(19)** (1949), 119–124 (1950).

Vidal, Jésus-Marie Tharrats. Fondements d'une mécanique projective. C. R. Acad. Sci. Paris **232**, 2397–2398 (1951).

Haacke, Wolfhart. Bemerkungen zur Stabilisierung eines physikalischen Pendels. I. Z. Angew. Math. Mech. **31**, 161–169 (1951). (German. English, French, and Russian summaries)

Known properties of the solutions of the Mathieu differential equation are used in a discussion of the motion of a simple pendulum with the point of support subjected to a vertical displacement varying periodically with the time. The discussion confirms and clarifies results obtained previously by Erdélyi [same Z. **14**, 235–247 (1934); **16**, 171–182 (1936)] and Klotter [Forschung Gebiete Ingenieurwesens. Ausg. B **12**, 209–225 (1941)]. The paper includes a study of the case in which the elasticity of the pendulum rod is taken into account, so that the system has two degrees of freedom. In this case the properties of the system with regard to stability may differ essentially from those in the case in which the elasticity of the rod is neglected.

L. A. MacColl (New York, N. Y.).

Matuzawa, Takeo. Über die Bewegung eines Punktsystems. Bull. Earthquake Res. Inst. Tokyo **22**, 130–139 (1944). (German. Japanese summary)

The paper is concerned with the propagation of earthquake waves considered as a tranfer of motion along a connected system of similar bodies. This problem, though by a somewhat different method, has been treated by Lord Rayleigh [Philos. Mag. (5) **44**, 356–362 (1897); The Theory of Sound, 2d ed., vol. 1, MacMillan and Co., London and New York, 1894, chapt. 6]. Applications to the "cut-up" earth crust and to the diffusion problem are indicated.

E. Leimanis (Vancouver, B. C.).

Bucerius, Hans. Der freie Fall auf der rotierenden Erde. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1950**, 77–83 (1951).

Let ω be the constant angular velocity of rotation of a sphere with radius R about an axis through the centre, and let at a point of this sphere with latitude φ, be given a rectangular coordinate system (ξ,η,ζ), rigidly connected with this rotating sphere and having the following orientation of the axes: the ξ-axis is directed to the south along the meridian, the η-axis direction is eastward along the parallel, both tangential to the sphere, and the ζ-axis direction is vertically upward to the zenith. Then the well-known equations (E) for the relative motion with respect to the (ξ,η,ζ)-coordinate system can be written down. In the application to the Earth, owing to its oblate shape (geoid) due to rotation, the constant terms containing the factor $R\omega^2$ must be omitted in these equations (E) and $-g$ replaced by $-\bar{g}$, the acceleration due to the resultant of gravitation and centrifugal force caused by rotation of the Earth.

Usually the equations (E) are integrated under the simplifying assumption that the centrifugal terms, containing ω^2 and small in comparison with the Coriolis force, are omitted. The author shows that these equations are also strongly integrable when the centrifugal terms are retained. The integration is accomplished with the initial conditions corresponding to free fall at height h. Since ωt is usually very small (of the order of time of fall/day), the coordinates ξ, η and ζ can be developed in power series of this small quantity. Comparison with the corresponding formulas of the approximate integration show that the differences are, as it is natural to expect, second order terms in ω. The correction term containing the factor $h\omega^2$ is missing in the expression for η (deviation to the East) in the approximate integration. This strong solution may be of practical interest for motions in constant gravitation fields when the time of fall t is an appreciable fraction of a day (rockets). The strong solution of the motion in constant gravitation fields is also given for more general initial conditions. *E. Leimanis.*

Drenick, R. The perturbation calculus in missile ballistics. J. Franklin Inst. **251**, 423–436 (1951).

The effect of disturbing factors on the flight of a rocket is considered by means of the calculus of perturbations. The chief problem considered is that of choosing the instant of power cutoff for a missile such as the German V-2 rocket in such a way as to eliminate errors in range due to variations in such quantities as weight, lift and drag coefficients and specific impulse. On the supposition that mechanisms can be constructed to perform the necessary calculations and integrations it is shown that this could be done. After cutoff further errors can, of course, arise, such as those due to wind velocity and the presence of after-burning. A theoretical scheme to eliminate such errors by remote radio control is proposed. *R. A. Rankin.*

Hydrodynamics, Aerodynamics, Acoustics

Slezkin, N. A. On the differential equations of the motion of a gas. Doklady Akad. Nauk SSSR (N.S.) **77**, 205–208 (1951). (Russian)

The author observes that the usual continuity equation takes into account only the density changes due to the macroscopic velocities but not to diffusion, whereas in treating the energy transfer both the convective heat transfer and the molecular phenomenon of heat conduction are considered. He derives, by considering the mass, momentum and energy transfer through the surface of a small parallelepiped, a new set of differential equations for the motion of a gas. The equations are too complicated to be reproduced here. *L. Bers* (Los Angeles, Calif.).

Vallander, S. V. The equations of motion of a viscous gas. Doklady Akad. Nauk SSSR (N.S.) **78**, 25–27 (1951). (Russian)

L'auteur forme les équations de la dynamique des gaz en tenant compte des échanges calorifiques, des effets de la diffusion et des variations des quantités de mouvement d'un volume élémentaire; les processus chimiques sont négligés. Les équations indéfinies ainsi obtenues—différentes des relations classiques—sont complétées par les conditions aux limites, valables au contact du gaz avec une paroi solide fixe. *J. Kravtchenko* (Grenoble).

★Görtler, H., Karas, K., Sauer, R., Schiller, L., and Wieghardt, K. Applied Mathematics. Part III. Mathematical Foundations of Fluid Mechanics. The American Fiat Review of German Science, 1939–1946, vol. 5. The O. W. Leibiger Research Laboratories, Inc., Petersburg, N. Y., 1950. x+293 pp.

A translation of the survey articles in Naturforschung und Medizin in Deutschland, 1939–1946, Band 5 [Dieterich, Wiesbaden, 1948]. Reviews of the individual articles will be found under the names of their authors in these Rev. **11**, 221, 225, 226.

Imai, Isao. On Sneddon and Fulton's solution for the irrotational flow of a perfect fluid past two spheres. J. Phys. Soc. Japan **5**, 284–285 (1950).

Sneddon and Fulton [Proc. Cambridge Philos. Soc. **45**, 81–87 (1949); these Rev. **10**, 215] deal with the irrotational flow of a perfect incompressible fluid past two spheres by means of a formula of Weiss [same Proc. **40**, 259–261 (1944); these Rev. **6**, 191] for one sphere. The results of Sneddon and Fulton are a definite improvement over the results of Endo [Proc. Phys.-Math. Soc. Japan (3) **20**, 667–703 (1938)] and Mitra [Bull. Calcutta Math. Soc. **36**, 31–39 (1944); these Rev. **6**, 67] who have also considered the problem of two spheres. The author believes, however, that Weiss' formula is not applicable to two spheres as used by Sneddon and Fulton and at best is a good approximation to the two sphere problem when the two bodies are far enough apart. *A. Gelbart* (Syracuse, N. Y.).

Morgan, G. W. A study of motions in a rotating liquid. Proc. Roy. Soc. London. Ser. A. **206**, 108–130 (1951).

The problem of flow produced by the motion of solids in a rotating liquid is of theoretical interest because of several remarkable experimental results obtained by G. I. Taylor which to the present do not have a satisfactory explanation in theory. Among the simplifications made by the author and his predecessors in treating the problem is to take the fluid as incompressible and inviscid and to assume small relative motion, that is, to neglect nonlinear terms in the equations of motion referred to axes rotating with the constant angular velocity of the fluid about a fixed axis. The author attempts to explain how steady, slow, two-dimensional motion can be produced by a boundary condition which is three-dimensional (as observed in experiments performed by G. I. Taylor), by considering the flow history from the moment at which the disturbance in the flow is created from rest (relative to the rotating system). For the particular flows studied by the author the results are in agreement with Taylor's experiments, in that the flow is found to become steady and two-dimensional if the disturbance which causes it approaches a steady state. For example, in the disturbance due to a disk which moves along the axis of rotation of the fluid, starting from rest, he finds that the ultimate flow pattern will be one in which a cylindrical column of liquid of the same diameter as the disk will move along with it, no fluid crossing the surface of this column. The author's solutions are obtained by means of Laplace transforms. The work of Taylor, Proudman, Grace and Görtler on this subject are discussed in some detail. *D. Gilbarg* (Bloomington, Ind.).

Dolapčiev, Bl. A generalized method of definition of the stability of an arbitrarily situated vortex street. Doklady Akad. Nauk SSSR (N.S.) **77**, 985–988 (1951). (Russian)

Soit une rue de tourbillons à deux files parallèles (distantes de $2h$), décalées de $2d$ l'une par rapport à l'autre. Si $2l$ est la distance de deux tourbillons consécutifs d'une file, la condition nécessaire de stabilité de la configuration s'écrit, d'aprés l'auteur, $\sinh \mu\pi = \sin \lambda\pi$, où $\lambda = d/l$, $\mu = h/l$. Comme cas particuliers, on retrouve un critère de von Kármán et les conclusions relatives au cas symétrique: $\lambda = 0$. *J. Kravtchenko* (Grenoble).

Dolapčiev, Bl. Application of N. E. Kočin's methods to the investigation of the equilibrium conditions of two-parameter vortex streets. Doklady Akad. Nauk SSSR (N.S.) **78**, 29–32 (1951). (Russian)

L'auteur étudie la stabilité d'une rue de tourbillons plans (formée de deux files parallèles). Si les perturbations initiales sont quelconques, la configuration précédente est instable au sens absolu, même si la condition nécessaire de Kármán est remplie: car, on peut toujours particulariser les conditions initiales de manière que l'axe du système tourbillonnaire s'écarte indefiniment de sa position primitive en se déplaçant parallèlement a lui-même. L'auteur cherche alors les conditions de stabilité au sens restreint de Kotchine [C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 18–22 (1939); ces Rev. **2**, 26]; il est conduit ainsi à caractériser les configurations les moins instables. *J. Kravtchenko.*

Dolapčiev, Bl. The stability of vortex streets. Doklady Akad. Nauk SSSR (N.S.) **78**, 225–228 (1951). (Russian)

Développant ses études antérieures [voir les deux analyses précédentes] sur la stabilité des rues de tourbillons, l'auteur construit une perturbation initiale d'une rue de tourbillons en quinconce (vérifiant la condition de stabilité de Kármán) qui permet d'expliciter les équations des trajectoires (stables) des tourbillons du système. L'auteur tire de la quelques conclusions relatives à la stabilité des configurations intermédiaires dont il décrit les aspects successifs. *J. Kravtchenko* (Grenoble).

Lu, Hsih-Chia. On the surface of discontinuity between two flows perpendicular to each other. Eng. Rep. Nat. Tsing Hua Univ. 4, no. 1, 40–62 (1948).

This paper considers the surface of discontinuity produced when a uniform parallel stream flows past and perpendicular to a uniform cylindrical stream (such as occurs when there is flow from a submerged orifice into a stream). In case the speed of the cylindrical stream is large compared to that of the parallel flow, the author makes it plausible that the steady three-dimensional problem can be replaced approximately by an unsteady plane problem, the solution to which describes the motion in parallel planes perpendicular to the axis of the cylindrical flow. The author therefore concentrates on the following plane flow problem: Given an initial uniform plane potential flow past a circular profile containing a fluid at rest; to determine the ensuing motion of the profile if the latter is a slip surface across which pressure is continuous, and if the motion inside the contour is also a potential flow. This problem is treated numerically in two ways, by the method of undetermined coefficients based on a series expansion in time, and by a method based on stepwise computation of the velocity and motion induced by the vortex sheet which comprises the slip surface. Results of numerical calculation by both methods are shown.

D. Gilbarg (Bloomington, Ind.).

Kaufmann, W. Der zeitliche Verlauf des Aufspulvorganges einer instabilen Unstetigkeitsfläche von endlicher Breite. Ing.-Arch. 19, 1–11 (1951).

The author notes that in two-dimensional incompressible flow a free straight line segment discontinuity (a plane ribbon in three dimensions) having a given symmetric circulation distribution (assumed elliptical in subsequent calculations) is unstable and tends to curl up at the ends, finally developing into two equal and opposite vortices. The author's problem is to find the motion of the elements of this discontinuity line as a function of time. A rigorous hydrodynamical treatment of the motion of a free line of discontinuity is usually very difficult. The author avoids this by means of such plausible assumptions as that of the motion of the line being predominantly parallel to itself, and by some rather heroic approximations. Position and time are expressed in terms of an unknown function over an interval for which only a few end conditions are known. However, two alternate guesses as to its form lead to substantially the same results. A comparison with experiment is made.

E. Pinney (Berkeley, Calif.).

***Arhangel'skii, V. A. Raščety neustanovivšegocya dviženiya v otkrytyh vodotokah. [The Calculation of Unsteady Motion in Open Channels].** Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1947. 136 pp.

I. Unsteady flow in open channels. II. Differential equations for unsteady flow in prismatic channels. III. Method of characteristics. IV. Method of instantaneous states. V. Flow in the form of a broken wave. VI. Source data for calculations; examples. VII. Examples of problems requiring calculations of unsteady flow.

Table of contents.

Batchelor, G. K. Note on a class of solutions of the Navier-Stokes equations representing steady rotationally-symmetric flow. Quart. J. Mech. Appl. Math. 4, 29–41 (1951).

The author considers a class of steady rotationally symmetric solutions of the Navier-Stokes equations describing (a) the flow in a half-space of an incompressible fluid rotating with uniform angular velocity γ at infinity and bounded by an infinite disc rotating with uniform angular velocity ω about the axis of symmetry, and (2) the flow between two parallel infinite discs rotating about the same axis with uniform angular velocities γ and ω. The flow problems are not solved explicitly, but are reduced to the solution of two nonlinear ordinary differential equations, which are not examined in any detail. The author's qualitative discussion of the dependence of the flows on the ratio γ/ω is the main portion of the paper. The above flows are a generalization of those considered by von Kármán [Z. Angew. Math. Mech. 1, 233–252 (1921)], in which there is no rotation of the fluid at infinity. Essentially the author's equations and a further generalization are given by R. Berker [Sur quelques cas d'intégration des équations du mouvement d'un fluide visqueux incompressible, Taffin-Lefort, Paris-Lille, 1936, pp. 29–30, 96–100].

D. Gilbarg.

Stewartson, K. On the impulsive motion of a flat plate in a viscous fluid. Quart. J. Mech. Appl. Math. 4, 182–198 (1951).

The motion in the boundary layer which arises when a semi-infinite plate is impulsively started from rest with velocity U is investigated. Both Rayleigh's and momentum-integral methods show that the solution is independent of x, the distance along the plate from the leading edge, when $Ut<x$ and depends on x as well as t when $Ut>x$. At $Ut=x$ the solution has an essential singularity, and as $t\to\infty$ the influence of t dies out exponentially. The nature of these singularities at $Ut=x$ and $t\to\infty$ has been obtained by studying the behavior of the solution of the Prandtl boundary layer equations. It is shown that the singularity at $Ut=x$ is different from that at $t\to\infty$ in that the latter is independent of how it is approached while the former is so dependent. Finally, the problem for compressible fluid is also discussed.

Y. H. Kuo (Ithaca, N. Y.).

Meksyn, D. Motion in the wake of a thin plate at zero incidence. Proc. Roy. Soc. London. Ser. A. 207, 370–380 (1951).

By a method developed by the author [Proc. Roy. Soc. London. Ser. A. 201, 268–278 (1950); these Rev. 12, 60], the motion in the wake of a flat plate without pressure gradient is studied. It is shown that for finite distance x from the trailing edge of the plate, if the equation of motion is solved by successive approximations and the nth approximation vanishes exponentially with the coordinate normal to the plate, then the $(n+1)$th approximation will also vanish exponentially. By carrying out this procedure to the second order, the author obtains an axial velocity different from Goldstein's earlier result [ibid. 142, 545–573 (1933)]. By a slightly different method, the solution in the neighborhood of the trailing edge is also given.

Y. H. Kuo.

Nevzglyadov, V. G. A new method in the dynamics of a viscous fluid. Doklady Akad. Nauk SSSR (N.S.) 77, 573–576 (1951). (Russian)

The author sketches a method of approximate solution of the Navier-Stokes equations for flows around a bounded body for which there exists a stream tube C containing the body which is of relatively small diameter far before and far behind the body. For simplicity the author limits himself to 2-dimensional flow. Let C_0 be the contour of the body. Let $\mathbf{u}_0$ be the solution for the inviscid case. Let (φ, ψ) be curvilinear coordinates defined by the equipotential lines and stream lines, and let $\psi=\pm\lambda$ be C, $\psi=0$ contain C_0, and

φ_1 and φ_2 be the values of φ for the front and rear stagnation points. The author writes $\mathbf{u}=\mathbf{u}_0+\mathbf{v}$, substitutes in the Navier-Stokes equations, and linearizes in $\mathbf{v}$ (assuming $\mathbf{v}$ small "on the average" compared to $\mathbf{u}_0$), a procedure similar to Oseen's. Then $\mathbf{u}$ is approximated in the "boundary layer," $\varphi_1<\varphi<\varphi_2$, $|\psi|\leqq\lambda$, by a polynomial in ψ/λ. This solution is to be joined smoothly with the solutions of the linearized equations on the boundary C. Formulas for the force and moment on the body are given. There are no error estimates, and certain details were not clear to the reviewer. However, the author makes strong claims for the usefulness of the method over wide ranges of Reynolds number. [Cf. the following review.] *J. V. Wehausen.*

Nevzglyadov, V. G. The flow of a viscous fluid about a flat plate. Doklady Akad. Nauk SSSR (N.S.) **77**, 795–798 (1951). (Russian)

The author applies the method of the paper reviewed above to flow about a flat plate of finite length parallel to the stream. An asymptotic solution for large Reynolds number is given, and also a formula for the drag coefficient for "arbitrary" Reynolds number. For large Reynolds number this formula gives a value close to the Blasius solution, and for small values a solution of the form given by Harrison and Filon [cf. Lamb, Hydrodynamics, 6th ed., Cambridge Univ. Press, 1932, §343a] although not identical with their solution. There seemed to be some arbitrary choice of constants involved. *J. V. Wehausen* (Providence, R. I.).

Zaat, J. A. Revised methods for routine calculations of laminar and turbulent boundary layers of two-dimensional incompressible flows. Nationaal Luchtvaartlaboratorium, Amsterdam. Report **F. 79**, i+14 pp. (6 plates) (1950).

Wuest, Walter. Beitrag zur Entstehung von Wasserwellen durch Wind. Z. Angew. Math. Mech. **29**, 239–252 (1949). (German. English, French, and Russian summaries)

Lord Kelvin was the first to consider the production of water surface waves by wind as a hydrodynamic instability problem. He took the undisturbed velocities in air and in water to be uniform and neglected viscosities of the fluids and obtained a minimum wind velocity for the production of waves which was much higher than the observed one. H. Jeffrey proposed two theories [Proc. Roy. Soc. London. Ser. A. **107**, 189–206 (1925); **110**, 241–247 (1926)] of which the second one was acceptable according to modern fluid dynamics principles and took account of the viscous effects approximately. This theory gives a minimum unstable wind velocity close to the experimental results. Modern studies in the hydrodynamic instability of boundary layers indicate however the importance of the variable velocity in the original unperturbed flow due to boundary layer formation over the interface of air and water. The present paper is a new analysis of the hydrodynamic instability of such air and water boundary layers including the effects of viscosity, but not the effects of temperature gradients. The author uses the method and the results of W. Tollmien [Nachr. Ges. Wiss. Math.-Phys. Kl. Göttingen **1929**, 21–44] for the boundary layer over a flat plate. By having a boundary layer profile, the author introduces a new length into the problem: the boundary layer thickness or, nondimensionally, the boundary layer Reynolds number. Using linear profiles the author shows that the effect of viscosity is to eliminate unstable waves of small wave velocity. The effect of curvature in the velocity profile is considered next by approximating the "Blasius profile" in the laminar air boundary layer by straight line and parabola. This curvature effect is found to be small. [Reviewer's remark: It is worthwhile to note that if the wave periods are much longer compared with the characteristic time in the fluctuations of a turbulent boundary layer, the analysis of instability could be extended to turbulent profiles.] *H. Tsien* (Pasadena, Calif.).

Heisenberg, Werner. On stability and turbulence of fluid flows. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1291, 60 pp. (1951).

Translated from Ann. Physik (4) **74**(379), 577–627 (1924).

Lee, T. D. Difference between turbulence in a two-dimensional fluid and in a three-dimensional fluid. J. Appl. Phys. **22**, 524 (1951).

The author remarks that by transforming the two-dimensional vorticity equation to the corresponding spectral form one can obtain $-(\partial/\partial t)\int_0^\infty F(k)k^2dk=2\nu\int_0^\infty F(k)k^4dk$ since the term corresponding to stretching of vortex filaments is absent. Here $F(k)$ is the energy spectrum. By assuming that $F(k)\propto k^{-5/3}$ for $k_0<k<k_s$, where k_s/k_0 is fairly large, he arrives at a contradiction. In the 3-dimensional case this contradiction doesn't arise. *J. V. Wehausen.*

⁎von Kármán, Th., and Lin, C. C. On the statistical theory of isotropic turbulence. Advances in Applied Mechanics, vol. 2, edited by Richard von Mises and Theodore von Kármán, pp. 1–19. Academic Press, Inc., New York, N. Y., 1951. $6.50.

This is an expanded version of an earlier paper [Rev. Modern Physics **21**, 516–519 (1949); these Rev. **11**, 226]. After a brief introduction and survey of recent history in this field, the authors discuss similarity laws and, in some detail, their proposed decay law for the spectrum [cf. the cited review]. *J. V. Wehausen* (Providence, R. I.).

Truesdell, C. On the velocity of sound in fluids. J. Aeronaut. Sci. **18**, 501 (1951).

With reference to a recent series of notes concerning the velocity of sound [Morduchow, same J. **16**, 635 (1949); **17**, 180–181 (1950); Morkovin, ibid. **17**, 180 (1950); Millikan, ibid. **17**, 252 (1950); Munk, ibid. **17**, 376 (1950)] the author calls attention to relevant published work on the subject which was not mentioned by any of the authors.

Saito, Osamu, and Amemiya, Ayao. On the solution of differential equations of the two-dimensional steady flow of compressible fluid. J. Phys. Soc. Japan **5**, 201–202 (1950).

In a potential gas flow the density ρ is a given function of the speed q. Set

$$w=\int\frac{\rho dq}{q},\quad f(w)=\frac{1}{\rho^2}-\frac{d}{dw}\left(\frac{1}{\rho}\right).$$

If $F(w,\psi)$ satisfies the equation

$$F_{ww}F_{\psi\psi}-F^2_{w\psi}=f(w), \tag{1}$$

then $\phi=F_w$, $\theta=-F_\psi$ may be interpreted as the potential and inclination of the velocity vector, respectively, of a gas flow with stream-function ψ. Solutions of (1) are:

$$F=\psi\int(-f)^{\frac{1}{2}}dw+\alpha f+G(w)$$

where α is an arbitrary constant, G an arbitrary function,

$$F=(\psi+C)^2(Aw+B)^{-1}+\frac{1}{2}\int dw\int(Aw+B)fdw$$

where A, B, C are arbitrary constants. A detailed presentation will appear elsewhere. *L. Bers.*

Martin, Monroe H. Steady, rotational, plane flow of a gas. Amer. J. Math. **72**, 465–484 (1950).

The author discusses steady rotational gas flows in two dimensions in a curvilinear coordinate system formed by streamlines and isobars (lines of constant pressure). (Flows for which streamlines are isobars are excluded.) This leads to the system of differential equations: (1) $u_p=y_\psi$, $v_p=x_\psi$, $uy_p-vx_p=0$, $uy_\psi-vx_\psi=\rho^{-1}$, where x, y are the Cartesian coordinates, u, v the velocity components, ψ the stream function, p the pressure and ρ the density. If $\partial(x,y)/\partial(p,\psi)\neq 0$, a solution of (1) furnishes a solution of the Euler equation and the continuity equation. To obtain a determinate system one adds to (1) the equation of state: $\rho=f(p,S)$, where S is the entropy, and specifies the entropy distribution by prescribing $S=S(\psi)$. By Bernoulli's theorem the speed q is a function of p and ψ. The flow is subsonic (supersonic) if $q_{pp}<0$ $(q_{pp}>0)$, potential if $q_\psi\equiv 0$. If the "Bernoulli function" $q=q(p,\psi)$ is given, a flow can be obtained by solving the quasi-linear equation

$$q\{(q_{pp}-q\theta_p^2)/\theta_\psi\}_p+(q^2\theta_p)_p=0 \tag{2}$$

for the function $\theta=\theta(p,\psi)=\arctan(v/u)$. This excludes flows for which $\theta_\psi\equiv 0$ (isoclinic flow). The author derives the most general equation of state admitting isoclinic flows. If this equation is of the form $\rho=\Sigma(S)\Pi(p)$, then all isoclinic flows are either Prandtl-Meyer flows (that is, have a one-dimensional hodograph) or are obtained from such by a device due to Munk and Prim [Naval Ordnance Laboratory Rep. **NOLM 9281**, 1947]. Finally, flows with $\theta=\theta(\psi)$ (rectilinear flows) are examined in detail.

L. Bers (Los Angeles, Calif.).

Dörr, J. Beitrag zu einer Wirbeltheorie des kompressiblen Mediums. Ing.-Arch. **18**, 378–384 (1950).

The author develops C. Possio's integral equation for a two-dimensional, oscillating airfoil in subsonic, compressible flow [Aerotecnica **18**, 441–458 (1938)] by introducing vortex concepts and following an analysis similar to that of L. Schwarz [Luftfahrtsforschung **17**, 379–386 (1940); these Rev. **3**, 286] for the incompressible problem.

J. W. Miles (Auckland).

Imai, Isao. An approximate method of calculating compressible fluid flow past a thin aerofoil. J. Phys. Soc. Japan **3**, 346–351 (1948).

Imai, Isao. On a new method of approximation for treating compressible fluid flow. J. Phys. Soc. Japan **3**, 352–356 (1948).

These two papers present two closely related approximate methods for computing nearly parallel subsonic gas flows past airfoils. Let $u=q\cos\theta$, $v=q\sin\theta$ be the velocity components, c the local speed of sound and $\mu^2=1-q^2/c^2$. First method: Assume that v is small compared to u, so that $(q/c)^2$ may be replaced by $(u/c)^2$, and c may be treated as a function of u. Neglecting small terms and setting $\omega=\int\mu du$, (1) $\xi=x$, $\eta=\int\mu dy$, it turns out that $\omega+iv$ is an analytic function of $\xi+i\eta$. In order that (1) have a meaning, an assumption for μ inside the profile must be made. The flow problem is now reduced to a boundary value problem for analytic functions. An iteration scheme involving the familiar cot-integral is used for solving this problem. (No convergence proof is given.) Second method: Using the assumption that θ is small and setting $\tau=\int\mu d\log q$ it is shown that $\theta+i\tau$ is (approximately) an analytic function of $\xi+i\eta$. If one sets $\mu=1$ inside the profile, it turns out that a solution of an incompressible flow problem yields at once a solution for the approximate problem considered. For points on the profile a "universal" velocity correction table can be given. Both methods are illustrated by numerical examples. For supersonic flows the same methods lead to the wave-equation. [As in all thin wing theories the basic assumptions break down near a stagnation point, but the author does not discuss this difficulty.] *L. Bers.*

Reissner, Eric. Note on the relation of lifting-line theory to lifting-surface theory. J. Aeronaut. Sci. **18**, 212–214 (1951).

The author discusses an approximate solution to the integral equation for a subsonic, lifting surface of infinite span with a (spanwise) periodic distribution of incidence by expanding in descending powers of the wave length s, which parameter is representative of the effective aspect ratio for a finite wing. The chordwise integral of the pressure is compared with the corresponding results predicted by the lifting line theories of Prandtl and Weissinger. It is found that both yield a correct determination of the coefficient of s^{-1}, but the Weissinger theory yields, in addition, an approximation to the coefficients of terms of order $s^{-2}\ln s$. The author points out that it is not possible to generalize the result that lifting line theory is the correct limiting form of lifting surface theory, citing the semi-infinite rectangular surface to prove his point. *J. W. Miles* (Auckland).

Reissner, Eric. On the theory of oscillating airfoils of finite span in subsonic compressible flow. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **1002**, 9 pp. (1950).

Issued earlier as Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1953** (1949); these Rev. **11**, 273.

Reissner, Eric. Extension of the theory of oscillating airfoils of finite span in subsonic compressible flow. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2274**, 16 pp. (1951).

The author proposes a method of solving the approximate integral equation for an oscillating rectangular airfoil in subsonic compressible flow, following the formulation of his earlier paper on the same subject [same Tech. Notes, no. **1953** (1949); these Rev. **11**, 273]. The principal difficulty appears in the involved nature of the kernel to the auxiliary integral equation for the spanwise distribution function, and further progress awaits the numerical tabulation of this kernel. Some consideration of the kernel is given in an appendix by Z. Kopal. *J. W. Miles* (Auckland).

Lomax, Harvard, and Sluder, Loma. Chordwise and compressibility corrections to slender-wing theory. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2295**, 44 pp. (1951).

Approximate solutions to the linearized, lifting surface problem are obtained by assuming the spanwise lift distribution to be elliptic and solving the resulting integral equation numerically for rectangular and triangular wings in both subsonic and supersonic flow. The results constitute an extension of the slender wing theory [Jones, Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **835** (1946); these Rev. **11**,

698]. The reviewer remarks that the theory on slender wing-bodies given by reference 3 [J. R. Spreiter, Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1662** (1948)] is incorrect, the correct analysis having been given by G. N. Ward [Quart. J. Mech. Appl. Math. **1**, 225–245 (1948); these Rev. **10**, 77], being valid for larger values of the effective aspect ratio. However it should be noted that the author's approximation differs from the correct, two dimensional result by a factor of $\frac{1}{4}\pi$ in the limit of infinite aspect ratio and for this reason is inferior to a related analysis given by Lawrence [Cornell Aero. Lab. Rep. **AF-673-A-1** (1950)] for the subsonic problem, which predicts the correct limits for both zero and infinite aspect ratio, albeit not applied explicitly to the triangular wing. In the case of the low aspect ratio supersonic rectangular wing a simpler and more accurate analysis has been given by K. Stewartson [Proc. Cambridge Philos. Soc. **46**, 307–315 (1950); these Rev. **11**, 699]. Nevertheless, the paper includes many interesting, numerical results not previously available. *J. W. Miles.*

Behrbohm, H., und Oswatitsch, K. Flache kegelige Körper in Überschallströmung. Ing.-Arch. **18**, 370–377 (1950).

The authors first obtain integral expressions for the perturbation velocities due to conical source distributions in the (x, y)-plane inside or outside the nose Mach cone; i.e., distributions constant along rays through the origin. These can be used to determine the flow about thin wings having conical symmetry; this is the method used by Puckett [J. Aeronaut. Sci. **13**, 475–484 (1946); these Rev. **8**, 109]. Here the method is extended to treat two special cases of asymmetrical (lifting) wings: (i) supersonic leading edge, side edge in stream direction, (ii) symmetrical planform with subsonic leading edges. Other wings can be obtained by Lorentz transformation of these. In both cases the integral equation for the source loading in the off-wing region is reduced to the form of a singular one familiar in aerodynamics [e.g., Söhngen, Math. Z. **45**, 245–264 (1939)]. The flat-plate cases of these two planforms are worked out in detail.

The authors have neglected to point out that the Lorentz transformation cannot convert a subsonic leading edge or side edge into a trailing edge, in the sense that the Kutta-Joukowski condition will not be satisfied. Thus, a class of interesting wings is excluded from their theory, which is otherwise very general. Presumably the extension could be carried out, but it might involve more complicated integral equations. *W. R. Sears* (Ithaca, N. Y.).

Mitchell, A. R., and Rutherford, D. E. Application of relaxation methods to compressible flow past a double wedge. Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 139–154 (1951).

Fox and Southwell [Proc. Roy. Soc. London. Ser. A. **183**, 38–54 (1944)] and others have conjectured that some supersonic flow calculations by relaxation methods may converge slowly or not at all since small corrections may produce very large changes in residuals. The authors report that in certain mixed sub- and supersonic flow calculations the changes in residuals have not been found to be excessive except in the immediate vicinity of the sonic line, where corrections must be chosen by trial and error to keep one of the functions involved in the calculations real-valued while minimizing the residuals. They present constant Mach number contours calculated with an equilateral triangular grid for symmetrical plane irrotational flow at $M=0.205$ past a rhomboidal airfoil with 60° angle at the leading edge in a channel five times as wide as the airfoil. Starting from a symmetrical first approximation, the relaxation calculations actually make it clear that the flow must be asymmetrical, and the final results contain a locally supersonic region near the maximum ordinate of the airfoil. No theoretical or experimental check of the accuracy of the results has been made, need for which is shown by the failure of the computed sonic line to pass through the maximum ordinate. No effort seems to have been made to detect signs of the breakdown of potential flow in a locally supersonic region adjacent to a straight boundary, predicted by Nikolsky and Taganov [Akad. Nauk SSSR. Prikl. Mat. Mech. **10**, 481–502 (1946); these Rev. **8**, 237; for English translations see these Rev. **10**, 639]. *J. H. Giese* (Havre de Grace, Md.).

Seeger, R. J., and Polachek, H. On shock-wave phenomena: waterlike substances. J. Appl. Phys. **22**, 640–654 (1951).

Same as Rep. **NOLR-1135**, pp. 37–82 (1950), Symposium on shock-wave phenomena, Naval Ordnance Laboratory, White Oak, Md.; these Rev. **12**, 216. *D. Gilbarg.*

Stewartson, K. On the interaction between shock waves and boundary layers. Proc. Cambridge Philos. Soc. **47**, 545–553 (1951).

For a weak shock wave of small strength ϵ, the pressure change across the shock is very gradual. In fact the pressure gradient is sufficiently small to allow the use of the conventional boundary layer theory to calculate the interaction of such a weak normal shock with the boundary layer. The author studies specifically the boundary layer interaction over a flat plate, using his own transformation for compressible flow [Proc. Roy. Soc. London. Ser. A. **200**, 84–100 (1949); these Rev. **11**, 553]. The undisturbed flow is then the "Blasius solution." By retaining only the lowest order terms in the perturbation equation, the equation turns out to be a linear equation. Applying the method of Laplace transform, the author determines, to the first order of approximation, the velocity gradient at the surface of the plate, and shows that separation of boundary layer occurs if the strength of the shock ϵ is $O(R^{-2/5})$ where R is the Reynolds number along the plate at the point of incidence of the shock. The paper concludes with a detailed study of the consistency of the approximations involved in the calculation and with the conclusion that the conventional boundary layer theory can be used provided no separation has occurred. *H. Tsien* (Pasadena, Calif.).

Chang, Chieh-Chien. Applications of von Kármán's integral method in supersonic wing theory. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2317**, 71 pp. (1951).

Von Kármán's Fourier integral method [unpublished (1946); J. Aeronaut. Sci. **14**, 373–402 (1947); these Rev. **9**, 111] is developed and applied to supersonic wing problems of the first kind (boundary conditions not mixed), explicit results being given for the thickness drag of a family of sweptback wings of taper ratios 0.2 and 0.5 and for the downwash distributions due to constant lift distributions over sweptback, tapered wings. The author appears to be unaware of the very extensive work on this same subject by H. R. Lawrence and his colleagues at Northrop Aircraft, Inc. [Rep. **GM-106**, **A-79**, Parts I, II, III (1947)] and of the related work of the reviewer [North American Aviation Rep. **AL-801** (1948); J. Aeronaut. Sci. **16**, 252–253 (1949)].

The reviewer believes that the author's remark (p. 2) that a general solution to the problem of the second kind ("direct" problem) can be effected via reduction to an integral equation of the Wiener-Hopf type is overly optimistic, this method being, at best, applicable only to wings with one subsonic and one supersonic leading edge. *J. W. Miles.*

Froehlich, Jack E. Nonstationary motion of purely supersonic wings. J. Aeronaut. Sci. **18**, 298–310 (1951).

The author considers first the reduction of the three-dimensional problem to an equivalent, two-dimensional problem, using an analysis suggested by Lagerstrom [cf. J. W. Miles, J. Aeronaut. Sci. **16**, 568–569 (1949); these Rev. **11**, 273]. The results are applied to the plunging, pitching and rolling oscillations of a rigid delta wing having supersonic leading edges, and to an oscillating flap on such a wing. Numerical results are given in the form of phase plots for the lift and coefficients at Mach numbers of 1.25 and 2. The author states that his results are in agreement with those obtained by the reviewer [U. S. Naval Ordnance Test Station, Inyokern, Calif., Tech. Memo. **RRB-36** (1950)]. *J. W. Miles* (Auckland).

Stewart, H. J., and Li, Ting-Yi. Source-superposition method of solution of a periodically oscillating wing at supersonic speeds. Quart. Appl. Math. **9**, 31–45 (1951).

This paper attempts to extend the validity of Evvard's "equivalent area" concept [Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1382** (1947); these Rev. **8**, 610] to harmonic motion of wing tips in supersonic flow. However, in the opinion of the reviewer, neither the extension nor the authors' analysis is correct, the evidence con being: (1) Application of the result to the rectangular wing leads to apparently incorrect results [Stewart and Li, J. Aeronaut. Sci. **17**, 529–539 (1950); these Rev. **12**, 453]. (2) As demonstrated by the authors, the result is not valid for "unit step" motion, whereas a correct result for harmonic motion certainly must be, in consequence of the linearity of the basic equations and Fourier's theorem. The authors' remark that the difficulty lies in the improper behaviour of the spectral integral at infinite frequency is appropriate only to the initial response, since for finite time the exact result may be approximated uniformly by a finite spectrum. (3) Sufficient conditions for the term by term equivalence of Eq. (25b) are not established, the statement that the expansions are power series in M^{-1} being incorrect due to the M-dependence of the coefficients of M^{-2n}. (In particular, the transformation to Mach coordinates is M-dependent.) It appears to the reviewer that the "equivalent area" concept is applicable to unsteady flow only when the motion is sufficiently slow to justify neglecting the square of the reduced frequency, thereby rendering possible reduction to an equivalent steady flow [J. W. Miles, J. Aeronaut. Sci. **16**, 378–379 (1949); these Rev. **10**, 755]. It is for this reason that Evvard obtained the correct result for uniform vertical acceleration of a rectangular wing [Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1699** (1948); these Rev. **10**, 78], albeit the general validity of the same paper (a copy of which is not presently available to the reviewer) may be suspect. *J. W. Miles* (Auckland).

Miles, John W. Transient loading of wide delta airfoils at supersonic speeds. J. Aeronaut. Sci. **18**, 543–554 (1951).

This paper was reviewed in report form [Naval Ordnance Test Station, Inyokern, Calif., Tech. Memo. **RRB-37** (1950); these Rev. **12**, 217].

Dorrance, William H. Nonsteady supersonic flow about pointed bodies of revolution. J. Aeronaut. Sci. **18**, 505–511, 542 (1951).

Jarre, Gianni. Moto di un fluido compressibile in una girante radiale di turbomacchina. Aerotecnica **31**, 77–81 (1951).

Betz, Albert. Strömungserscheinungen in umlaufenden Schaufelkanälen. Aerotecnica **31**, 25–28 (1951).

Goldstein, Arthur W. Axisymmetric supersonic flow in rotating impellers. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2388**, 36 pp. (1951).

Naylor, V. D. The critical flow of a gas through a convergent nozzle. A simple and more comprehensive treatment of throat conditions in critical nozzle flow. Aircraft Engrg. **23**, 160–162 (1951).

Eady, E. T. Long waves and cyclone waves. Tellus **1**, no. 3, 33–52 (1949).

This paper outlines the author's ambitious program for the development from first principles of a quantitative theory of the genesis and development of wave cyclones and long waves, and develops the basic equations of the first admittedly crude models. More specifically, the paper is concerned only with initially convectively stable systems (which remain so after small perturbations) in laminar frictionless adiabatic motion of an incompressible rotating baroclinic fluid. A further distortion of the models as compared to atmospheric flow is introduced by the use of a rectangular coordinate system as a frame of reference for large scale phenomena.

The first set of models is constructed with the additional assumptions of constant coriolis parameter, uniform motion at each level, and constant vertical shear (thermal wind) and static stability. The resultant fifth-order differential equation of the perturbed motion can be reasonably simplified to an ordinary second order equation which has a consistent set of solutions which are periodic in x, y, and t and with amplitudes initially a function of height only. The first set of boundary conditions to be applied to these solutions defines a fluid of infinite horizontal extent in which motion takes place between two horizontal rigid plane boundaries. The resultant model has the following properties: 1) the disturbances are, at each level, a series of growing ridges and troughs with their axes normal to the unperturbed thermal wind; 2) the waves travel with the mean unperturbed current; 3) the wavelength is determined by the parameters of the unperturbed system, and, if the distance between the rigid boundaries is taken to be about the height of the tropopause, the wavelength of the disturbance of maximum growth-rate is approximately that of observed long waves in the westerlies; 4) for smaller vertical extents, the dominant wavelength is of the order of magnitude of observed extra-tropical cyclones; 5) the pressure trough slopes upwards and backwards in the atmosphere, while the warm tongue slopes upwards and forwards; 6) at low levels the warm tongue is slightly ahead of the pressure trough but at high levels the warm tongue is slightly to the rear of the upper pressure ridge; 7) upward motion is at a maximum at middle levels one-eighth wavelength ahead of the surface pressure trough; 8) motion is upward toward cold air and downward toward warm air; and finally 9) there is a general

decrease in the potential energy of the system as a result of the disturbance.

Modifications of the above basic model are obtained by varying the number and height of horizontal boundaries and the vertical distribution of static stability. In each case orders of magnitude and behavior in agreement with those of observed atmospheric systems are obtained. Another important set of modifications is obtained when the assumption of infinite horizontal extent is replaced by a set of side-by-side regimes. From the large number of possible combinations of vertical and side-by-side regimes, it is easy to select ones which are in good agreement with observation. The principal results here concern the development of discontinuities of wind, pressure gradient, and entropy as realizations of a latent discontinuity (in static stability) between saturated and unsaturated air. These discontinuities are more complex than the usual "fronts," and are a result rather than a cause of the development of the original perturbation. Another important feature of the more complicated cases is the loss of symmetry, troughs being accentuated and ridges flattened at all levels.

The second set of models takes into account the variation of the coriolis parameter with latitude. The principal modification in the behavior of the models is a lowering of the steering level which, in the symmetrical cases of the first set of models, was the middle level of the system. It is also shown that the lowering is of a greater order of magnitude than that which results when the compressibility of the atmosphere is taken into account. The baroclinic model investigated by Charney [J. Meteorol. **4**, 135–162 (1947); these Rev. **9**, 163] falls naturally into this set, except for the incompressibility assumption of the present paper, which introduces only a distortion rather than a difference in kind. Even so, the results are not in every respect in agreement with those of Charney. The paper concludes with an analysis of the energy changes associated with various types of "overturning" and with a discussion of the implications of the results of the paper as a whole with regard to the ultimate limitations of weather forecasting. *W. D. Duthie.*

Nagel, H. Zur Stabilitäts-Theorie des geostrophischen Windes. Arch. Meteorol. Geophys. Bioklimatol. Ser. A. **3**, 229–267 (1 plate) (1951).

The motion of a "disturbed" element of air is considered whose trajectory always lies in the neighbourhood of the trajectory of an element moving geostrophically. The two elements move in the same pressure and temperature fields and have the same potential temperature. The criteria of stability for the disturbed element are shown to depend on the sign of the second variation of the total energy of the element in geostrophic motion. Various expressions for this second variation are given, both for middle and high latitudes and for equatorial regions. Conditions for stable, unstable and neutral motion are listed and numerical details are given for the case of latitude 50°. The connection between the inclination of the isentropic surfaces and the different kinds of stability and instability is also worked out.

G. C. McVittie (London).

Truesdell, C. Correction to the paper "On the effect of a current of ionized air upon the earth's magnetic field." J. Geophys. Res. **56**, 134 (1951).

See same J. **55**, 247–260 (1950); these Rev. **12**, 568.

Vogel, Théodore. Vibrations des espaces clos à parois déformables élastiques. J. Phys. Radium (8) **11**, 627–632 (1950).

The author first points out the necessity of using wave rather than geometric theory for the discussion of room acoustics. He then goes on to discuss the effect of elastically deformable walls by perturbation theory. *H. Feshbach.*

Levitas, Alfred, and Lax, Melvin. Scattering and absorption by an acoustic strip. J. Acoust. Soc. Amer. **23**, 316–322 (1951).

The problem discussed in this paper is that of the scattering and absorbtion of sound by an infinitely long strip of material placed on an infinite hard plane wall. The strip is characterized by a normal acoustic impedance that may possess a resistive component. The problem of scattering by an infinitely hard or infinitely soft strip was treated by Morse and Rubenstein [Physical Rev. (2) **54**, 895–898 (1938)] who used expansions in Mathieu functions after a transformation of the wave equation to ellipsoidal coordinates. For intermediate impedances, the problem does not separate, but leads to an infinite set of equations for the expansion coefficients. This method was used by Pellam [J. Acoust. Soc. Amer. **11**, 396–400 (1940); these Rev. **2**, 28].

In the present paper techniques which have proved valuable in diffraction theory are applied to the problem. The wave equation is reformulated as an integral equation for the pressure on the strip. This integral equation is of generalised Wiener-Hopf type with integration over a finite interval, a type for which no general solution is known. However, the pressure on the strip is of no great physical importance. Without solving the integral equation, it is possible to deduce from it a variational expression for the amplitude of sound scattered in any direction. All quantities of physical interest (scattering amplitudes and cross-sections) can be obtained fairly accurately if one uses rough approximations for the unknown pressure on the strip. Comparison with Pellam's results show that the variational methods have an accuracy of better than 2% over the entire frequency range.

E. T. Copson (St. Andrews).

Elasticity, Plasticity

Lodge, A. S. On the use of convected coordinate systems in the mechanics of continuous media. Proc. Cambridge Philos. Soc. **47**, 575–584 (1951).

The author insists upon the fact that it is possible to write all equations governing the motion of a continuous medium in a form such that time and "convected" (i.e. Lagrangian, material) coordinates are the only independent variables. While in general such equations would be extremely elaborate, he gives an example concerning a visco-elastic material in an oscillating cup which he is actually able to solve explicitly. *C. Truesdell.*

Aržanyh, I. S. The integral equations of the deformation vector of the statics of an isotropic elastic body. Doklady Akad. Nauk SSSR (N.S.) **75**, 783–786 (1950). (Russian)

Let $u=(u_1, u_2, u_3)$ be the displacement vector of a three-dimensional isotropic elastic body occupying a bounded domain Q possessing a smooth boundary surface S. The displacement u satisfies the system of differential equations of equilibrium:

$$\mu\nabla^2 u+(\lambda+\mu)\nabla \operatorname{div} u+f=0,$$

where λ and μ are Lamé's constants of elasticity and f is the body force. Assuming that the domain Q possesses a Green's function G for the Dirichlet problem, the author obtains, in terms of G, an integral equation satisfied by the displacement u in the case of the first boundary value problem of elasticity, when u is prescribed on the boundary S. Analogously, assuming that the domain Q possesses a Green's function N for Neumann's problem, an integral equation satisfied by u, the kernel of the integral equation involving N, is obtained when u is a solution of the second boundary value problem of elasticity when the surface forces are prescribed on S. The derivation of the integral equations is based on two integral equations for the dilatation, div u, obtained previously [same Doklady (N.S.) **73**, 41–44 (1950); these Rev. **12**, 651]. *J. B. Diaz* (College Park, Md.).

Aržanyh, I. S. The integral equations of the dynamics of an elastic body. Doklady Akad. Nauk SSSR (N.S.) **76**, 501–503 (1951). (Russian)

Consider the problem of finding integral equations satisfied by the displacement u of a three-dimensional isotropic elastic body, which satisfies the system:

$$\mu\nabla^2u+(\lambda+\mu)\nabla \operatorname{div} u+lu=f,$$

where λ and μ are Lamé's constants, l is a parameter, and f is a given function. The displacement u is subjected either to the first boundary condition, displacement prescribed on the boundary, or to the second boundary condition, surface forces prescribed on the boundary. V. D. Kupradze [Boundary problems in the theory of vibrations and integral equations, Moscow, 1950] has given integral equations whose kernels involve the parameter l. The author [same Doklady (N.S.) **73**, 41–44 (1950); these Rev. **12**, 651] has given integral equations whose kernels involve the Green's functions for the Dirichlet and Neumann problems. In the present paper the author derives integral equations whose kernels do not depend on l or on the aforementioned Green's functions.

J. B. Diaz (College Park, Md.).

***Hlitčijev, J. Poglavlja iz teorije elastičnosti sa primenama. [Chapters from the Theory of Elasticity with Applications].** 2d ed. Naučna Knjiga, Belgrade, 1950. viii+216 pp. (Serbo-Croatian)

Szelągowski, Franciszek. Problème d'élasticité plane en fonctions de variables complexes. Arch. Méc. Appl., Gdańsk **3**, 45–51 (1951). (Polish. French summary)

The application of complex variables in the theory of elasticity was developed mainly by the mathematicians Kolosov [Application of Complex Variables to the Theory of Elasticity, Moscow, 1935] and Mushelišvili [Some Fundamental Problems of the Mathematical Theory of Elasticity, Moscow-Leningrad, 1949; these Rev. **11**, 626], who derived in a rather roundabout way formulas for the stresses and displacements in two-dimensional problems. The author of this paper derives them very simply using the Airy function and Goursat's biharmonic equation.

T. Leser (Lexington, Ky.).

Morris, Rosa M. The boundary-value problems of plane stress. Quart. J. Mech. Appl. Math. **4**, 248–256 (1951).

Following the complex potential method of solution of the two-dimensional anisotropic problems of plane elasticity developed by G. H. Livens and R. M. Morris [Philos. Mag. (7) **38**, 153–179 (1947); these Rev. **9**, 121], the author discusses the elastostatic anisotropic problem for an elliptical plate subjected to a prescribed stress distribution in the plane of the plate. The discussion is illustrated by an example of a plate subjected to a constant pressure and uniform shearing stress along the boundary, and by three other examples of certain stress distributions along the boundary which are not self-equilibrating. The solution of this difficult problem by a method closely similar to that of this paper was given by S. G. Lechnitzky [C. R. (Doklady) Acad. Sci. URSS (N.S.) **15**, 535–538 (1937)], and by a different method by P. P. Kufarev [ibid. **23**, 221–223 (1939)].

I. S. Sokolnikoff (Los Angeles, Calif.).

Prokopov, V. K. The deflection of a circular plate by an axially symmetric load. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 527–536 (1950). (Russian)

The deflection of thick circular plates, under uniformly distributed load applied normally to the faces of the plate, in the absence of either radial or axial displacements on the lateral surface of the plate, was considered by C. Clemmow [Proc. Roy. Soc. London. Ser. A. **112**, 559–598 (1926)] and, under a concentrated load at the center of a face of the plate and zero axial displacement on the lateral surface of the plate, by A. Nadai [Die elastischen Platten . . . , Springer, Berlin, 1925]. A. I. Lur'e [same journal **6**, 151–168 (1942); these Rev. **5**, 138] indicated a method for constructing solutions of the equations of elasticity and used his method for determining the deflection of a thick circular plate under a uniformly distributed load. In the present paper the author applies Lur'e's method to determine the deflection of a thick circular plate under arbitrary radially symmetric distributed load applied normally to the faces of the plate, the radial displacement being zero on the lateral surface. The solution is shown to yield the previous result for uniformly distributed load and, as limiting cases, for a concentrated load at the center, plus known results for thin elastic plates. *J. B. Diaz* (College Park, Md.).

Filippov, A. P. The deformation of an elliptic plate with simply supported boundary under the action of concentrated loads. Akad. Nauk SSSR. Inženernyĭ Sbornik **5**, no. 2, 71–82 (1949). (Russian)

Analytically, the boundary value problem considered in this paper consists in the determination of a real-valued function $w(x, y)$ (deflection of a plate) defined on an ellipse in the (x, y)-plane, such that: (a) in the interior of the ellipse w satisfies, save at a finite number of points of the interior (where concentrated loads normal to the plane of the plate are present), the partial differential equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=\frac{1}{D}f(x, y),$$

where $f(x, y)$ is a given function (the distributed load normal to the plane of the elliptic plate) and D is an elastic constant; at each of the exceptional set of points w behaves like $Cr^2 \log r$ plus a regular function, where C is a given real number (proportional to the intensity of the given concentrated load at the point) and r is the Euclidean distance from the point. (b) On the elliptical boundary w satisfies the conditions (simply supported plate) $w=0$,

$$M=-D\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)$$

$$-D(1-\nu)\left(\cos^2\theta\frac{\partial^2 w}{\partial x^2}+\sin^2\theta\frac{\partial^2 w}{\partial y^2}+\sin 2\theta\frac{\partial^2 w}{\partial x\partial y}\right)=0,$$

where D and ν are elastic constants and θ is the angle between the outer normal to the boundary and the positive x axis. The boundary value problem is first formulated in terms of elliptic coordinates. A formal solution is then given, expressed as an infinite series whose coefficients may be determined numerically by solving linear equations, in the special case of two concentrated loads, of equal magnitude and direction, symmetrically placed with respect to the major axis of the ellipse. The particular case of a concentrated load at the center of the plate is dealt with numerically.

J. B. Diaz (College Park, Md.).

Nash, W. A. Bending of annular elliptical plates loaded by edge moments. Bull. Calcutta Math. Soc. **42**, 189–198 (1950).

The problem of the small deflections of a thin plate bounded by two confocal ellipses and loaded by bending moments around both boundaries is presented. The faces of the plate are assumed free from load and the Lagrange differential equation for the deflection of the middle surface of the plate reduces to the biharmonic equation. Elliptic coordinates are used and an infinite series solution of the biharmonic equation is assumed. This solution is then made to satisfy the boundary conditions on the two confocal ellipses. Three types of edge conditions are considered involving prescribed deflections and bending moments. Each type leads to linear equations for the coefficients in the series solution. These equations are difference equations and no attempt has been made to solve them.

R. M. Morris (Cardiff).

√ **★Friedrichs, K. O. Kirchhoff's boundary conditions and the edge effect for elastic plates.** Proc. Symposia Appl. Math. vol. 3, pp. 117–124. McGraw-Hill Book Co., New York, N. Y., 1950. $6.00.

Let x, y, z be the coordinates of a point of an elastic plate bounded by the planes $z=\pm c$. The stresses according to the theory of "thin" plates, with Kirchhoff's boundary conditions, and the stresses according to the theory of "moderately thick" plates may be obtained by introducing the new variable $\zeta=z/c$, by assuming that $O(\partial/\partial\xi)=O(\partial/\partial x, \partial/\partial y)$ and by expanding the solutions of the equations of three-dimensional elasticity in powers of c cf.[J. N. Goodier, Trans. Roy. Soc. Canada. Sect. III. (3) **32**, 65–88 (1938)]. The solution thus obtained does not in general permit the satisfaction of the condition of a free edge where in a theory of "thin" plates one would like to satisfy three boundary conditions instead of Kirchhoff's two. The difference between the actual solution and the solution obtained above is known to be of importance only in an edge zone of the plate the width of which is $O(c)$. In order to calculate this difference the author introduces a new second type of series development in powers of c on the basis of three new variables $\xi=x/c$, $\eta=y/c$, $\zeta=z/c$. He indicates that the successive terms of the two types of developments cannot be determined independently of each other and presents some results so far obtained. Details are promised in a later publication.

E. Reissner (Cambridge, Mass.).

Owens, A. J., and Smith, C. B. Effect of a rigid elliptic disk on the stress distribution in an orthotropic plate. Quart. Appl. Math. **9**, 329–333 (1951).

Korenev, B. G. On the bending of unbounded plates lying on an elastic foundation. Doklady Akad. Nauk SSSR (N.S.) **78**, 417–420 (1951). (Russian)

The author considers a plate of uniform thickness described as in the above title. The properties of the elastic foundation are such that the pressure in one point and the deflection of the plate in another one are related by a function of the distance between those two points. The author calls this function the kernel function K. The formulas for deflection $w(x, y)$ in rectangular and polar coordinates are given without proof for various types of loads and foundations.

T. Leser (Lexington, Ky.).

Korenev, B. G. On the bending of a plate lying on an elastic foundation by loads distributed on a straight line and on a right angle. Doklady Akad. Nauk SSSR (N.S.) **79**, 411–414 (1951). (Russian)

Pozzati, Piero. Sulla risoluzione, in serie semplice, della lastra rettangolare appoggiata, sottoposta a un carico o a una coppia concentrati in un punto qualunque. Boll. Un. Mat. Ital. (3) **5**, 239–247 (1950).

Nowacki, W. Quelques cas particuliers de flambage des plaques. Arch. Méc. Appl., Gdańsk **2**, 107–122 (1950). (Polish. French summary)

The author considers two cases: 1. A plate of infinite length freely supported on the edges, compressed at the edges by a distributed load q in the plane of the plate, and loaded by a distributed load p perpendicular to the plane of the plate. The plate is compressed additionally by two concentrated forces P and also has points of support between the edges. 2. A rectangular plate freely supported on the perimeter, compressed at two opposite edges by a distributed load q in the plane of the plate, and loaded by a distributed load p perpendicular to the plane of the plate. The case where an additional point of support is added is also considered. The solutions of the partial differential equations in the form of infinite series are found from the given conditions, and the critical values of P and q in both cases are determined.

T. Leser (Lexington, Ky.).

Lur'e, A. I. On the equations of the general theory of elastic shells. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 558–560 (1950). (Russian)

In the classical theory of elastic shells the three-dimensional problem is reduced to a two-dimensional one through a specification of eight forces and moments which are statically equivalent to the distribution of the normal stresses σ_1 and σ_2 and the shear stresses $\tau_{12}=\tau_{21}$ along the lines of curvature in the middle section. Four of these quantities S_1, S_2, H_1, and H_2 are connected by the equation $S_1-S_2+H_1/R_1-H_2/R_2=0$ and are specially introduced to account for the shear stress τ_{12}. This note shows that in all equations of the elastic theory of shells these four quantities can be replaced by two quantities S and H given by:

$$2S=S_1+S_2-H_1/R_1-H_2/R_2,\quad 2H=H_1+H_2.$$

Further, within the limits of accuracy of the thin shell theory, S and H are given by:

$$S=\frac{Eh}{2(1+\nu)}\gamma,\quad H=-\frac{Eh^3}{12(1+\nu)}\omega^*$$

where γ and ω^* are respectively the shear strain and the rotation of the middle surface.

H. I. Ansoff (Santa Monica, Calif.).

Osgood, W. R., and Joseph, J. A. On the general theory of thin shells. J. Appl. Mech. 17, 396–398 (1950).

This paper concerns two corrections to Love's formulae for the small deformation of thin shells [Philos. Trans. Roy. Soc. London. Ser. A. **179**, 491–546 (1888)]. The first, originally observed by Lamb [Proc. London. Math. Soc. (1) **21**, 119–146 (1890), see especially pp. 135–136], concerns the formulae for changes of curvature κ_1, κ_2 and twist τ; Lamb's result, expressed in Love's notations, is that the changes of curvature are $k_1=k_1-\epsilon_1/R_1$, $k_2=\kappa_2-\epsilon_2/R_2$, while the twist is given by the symmetrical formula

$$t=\frac{1}{2B}\left[\frac{\partial}{\partial\beta}\left(\frac{u}{R_1}\right)-\frac{1}{R_2}\frac{\partial u}{\partial\beta}\right]+\frac{1}{2A}\left[\frac{\partial}{\partial\alpha}\left(\frac{v}{R_2}\right)-\frac{1}{R_1}\frac{\partial v}{\partial\alpha}\right]$$
$$+\frac{1}{AB}\left[\frac{\partial^2 w}{\partial\alpha\partial\beta}-\frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha}-\frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta}\right],$$

in place of Love's unsymmetrical τ. The authors repeat the observation of the referee of Lamb's paper [Lamb, loc. cit., p. 137, footnote] that Lamb's expressions are derivable from Love's general apparatus if one simply avoids "mistakes in the working."

[Reviewer's note: These results may be compared with those of R. Byrne [Univ. California Publ. Math. (N.S.) **2**, [No. 1, Seminar Rep. in Math. (Los Angeles)], 103–152 (1944), equations (21), (29), (39); these Rev. **5**,.250] who by a purely formal process introduces quantities $T_\alpha\equiv\kappa_1$, $T_\beta\equiv\kappa_2$, $T_{\alpha\beta}\equiv t+(1/R_1+1/R_2)\bar{\omega}/2$. Use of the authors' notations in place of Byrne's make Byrne's final equations assume a form nearer to the usual:

$$N_1=D(\epsilon_1+\sigma\epsilon_2)-K(1/R_1-1/R_2)k_1,$$
$$2N_{12}/(1-\sigma)=D\bar{\omega}+K(1/R_1-1/R_2)(t-[1/R_1-1/R_2]\bar{\omega}/2),$$
$$M_1=K[K_1+\sigma K_2+(\epsilon_1+\sigma\epsilon_2)/R_2],$$
$$2M_{12}/(1-\sigma)=K[2t+\bar{\omega}/R_2].$$

As was noted by Lamb [loc. cit., p. 124 and footnote, p. 136] and Novojilov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 160–164 (1943); these Rev. **5**, 139], the assumptions of shell theory make it uncertain whether differences such as ϵ_1/R_1 are significant. Cf. also the discussion of this point given by Hildebrand, Reissner, and Thomas [Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1833** (1949), especially pp. 25, 28; these Rev. **11**, 69].]

The second part of the authors' note concerns the equilibrium conditions. Here they claim that in Love's equations for moderately large deformations certain terms are omitted. [Reviewer's note: When the coordinates are referred to the deformed rather than the undeformed shell, the ordinary equilibrium equations of shell theory, without the added rotation terms of Love or the authors, are rigorously valid for deformations of any magnitude [Novojilov, loc. cit.; the reviewer, Trans. Amer. Math. Soc. **58**, 96–166 (1945), §§4, 5, 8; these Rev. **7**, 231]; see also the reviewer's discussion of this paper [same J. **18**, 231–232 (1951)].]

C. Truesdell (Bloomington, Ind.).

Zanaboni, Osvaldo. Distribuzione delle tensioni e delle deformazioni intorno ad un punto entro le lastre a doppia curvatura. Ann. Triestini. Sez. 2. (4) 3(19) (1949), 105–118 (1950).

Benischek, J. Allgemeine Berechnung der Spannungen in einem durch inneren Überdruck (p atü) belasteten und von aussen ungleichmässig erwärmten, kreisförmig gekrümmten Rohre. Österreich. Ing.-Arch. 5, 117–129 (1951).

The author summarizes the known power series solution of the differential equation for toroidal shells subject to axially symmetric loads [H. Wissler, Dissertation, Zürich, 1916] introducing appropriate loading terms for an axially symmetric temperature distribution. A practical application of the theory is mentioned but without numerical data.

R. A. Clark (Cleveland, Ohio).

Chien, Wei-Zang, and Ho, Shui-Tsing. Asymptotic method on the problems of thin elastic ring shell with rotational symmetrical load. Eng. Rep. Nat. Tsing Hua Univ. 3, no. 2, 71–86 (1948).

Problems in the small deflection theory of thin ring shells subject to rotationally symmetric loads are solved by a successive approximation method based on the assumption that all quantities may be expanded in powers of a small parameter $\lambda=a/r$, where a is the radius of the meridional cross section and r is the radius of the center line of the ring. The first approximation corresponds to the limiting case ($r\to\infty$) of an infinitely long cylindrical shell; consequently, the method applies only to problems involving loads for which such a cylindrical shell is in equilibrium. Two examples of edge loading are considered. Curves are given for the stresses and deflections for an example of an edge moment. [It has been the reviewer's experience that the present successive approximation method is valid not simply when λ is sufficiently small, as the authors seem to imply, but only when $\lambda(a/h)$ is sufficiently small (where h is the thickness of the shell). Since the theory of thin shells is valid only if $a/h\gg1$, this is a much stronger restriction. It is also felt that the combination $\lambda(a/h)=a^2/rh$ is a more natural parameter to use than λ. Solutions of ring shell problems are primarily characterized by the size of this combination. A variation in the value of λ alone ordinarily has only a secondary effect.]

R. A. Clark.

Wei, Chang. Der Spannungszustand in Kreisringschale und ähnlichen Schalen mit Scheitelkreisringen unter drehsymmetrischer Belastung. Sci. Rep. Nat. Tsing Hua Univ. Ser. A. 5, 289–349 (5 plates) (1949).

The present paper deals with axisymmetrical deformations of thin elastic toroidal shells. The object is similar to the work of R. A. Clark [J. Math. Physics **29**, 146–178 (1950); these Rev. **12**, 557]. The author obtains the asymptotic solution of the homogeneous differential equation in terms of Bessel functions of order $\frac{1}{3}$, not knowing of the Tables of the Modified Hankel functions of Order $\frac{1}{3}$ and their Derivatives [Annals of the Computation Laboratory of Harvard Univ., vol. 2, 1945; these Rev. **8**, 53]. The asymptotic particular solution of the nonhomogeneous equation which is the crucial part of the work appears in a form which appears to be more complicated than the corresponding result in Clark's work. As example of application the author calculates the stresses in a complete toroidal shell which carries equal and opposite line loads in the direction of the axis of the torus.

E. Reissner.

★Popov, E. P. Nelineĭnye zadači statiki tonkih sterżneĭ. [Nonlinear Problems of the Statics of Thin Rods]. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1948. 170 pp. (1 plate).

This is a self-contained monograph devoted to a systematic exposition of solutions of a broad class of elastostatic problems on large deflections of thin rods and flat blades. The theory, including a careful derivation of the Kirchhoff equilibrium equations, is contained in the first chapter (49 pp.). The remaining two chapters (103 pp.) deal with the applications of the theory to diverse problems on deflection and stability of thin rods deformed in one plane. The book summarizes all significant published results on problems of this category up to 1948. A distinguishing feature of the work is that the exact solutions are presented in a form suitable for numerical computation, so that they can be used directly by a stress analyst. The bibliography contains 85 items, the earliest of which is dated 1867.

I. S. Sokolnikoff (Los Angeles, Calif.).

Nowiński, J. Flexure of beams by terminal loads. Arch. Méc. Appl., Gdańsk **2**, 89–105 (1950). (Polish. English summary)

The author considers an isotropic homogeneous cantilever beam of uniform cross-section, fixed at one end, $z=0$, and loaded at the other, $z=l$, by forces equivalent to a single force P $(P_x, P_y, 0)$, acting in the point (x_W, y_W, l). The z-axis coincides with the central line of the beam, which is the locus of the centroid of a cross-section. The body forces are assumed to be zero. The displacements (u, v, w) are found from the general formula by substituting the constants defined from the above conditions. The author derives the expression for the angle of rotation of an element of a cross-section in the plane of this cross-section, which he calls the local rotation. The local rotation of an element containing the centroid is called the mean rotation. The local rotation depends on the fixing conditions and on the position of the load point. If the load point coincides with the position of the center of flexure the mean amount of rotation for every cross-section is constant, and generally different from zero. The mean amount of rotation may equal zero only if the fixed points are suitably chosen or if the beam is subjected to pure torsional couple. The axis of local pure translation which is the locus of a point which is only translated, and the axis of pure rotation which is the locus of a point which is not translated in the plane of a cross-section, are found. These loci, in general, are some space curves; when the beam is subjected to pure torsional couple the axis of pure rotation is the straight axis of twist.

T. Leser (Lexington, Ky.).

Pudovkin, M. A. On the computation of the axis of a bent beam. Doklady Akad. Nauk SSSR (N.S.) **77**, 993–995 (1951). (Russian)

The author considers a beam with arbitrary supports at the ends and arbitrary loads in one plane. The loads, if discontinuous, have a finite number of discontinuities n. By integrating n times the general differential equation of the bending curve and transforming the multiple integrals, the solution of the equation in terms of Stieltjes integral is obtained. The solution can also be expressed as a Volterra integral equation. The shearing force, the bending moment and the angle at the support can be evaluated from the solution by differentiation. *T. Leser* (Lexington, Ky.).

Sheng, P. L. Note on the torsional rigidity of semi-circular bars. Quart. Appl. Math. **9**, 309–310 (1951).

The author finds the exact value for the torque T transmitted by a semi-circular shaft of radius a, shearing modulus μ, and angle of twist α, to be

$$T=\mu\alpha a^4(\pi^2-8)/2\pi=0.297556\mu\alpha a^4$$

which is to be compared with published values $0.296\mu\alpha a^4$ or $0.2966\mu\alpha a^4$. The closed form of the result is found by obtaining exact sums of several series. *D. L. Holl.*

Gorgidze, A. Ya. The torsion and bending of composite bars with slightly curved axes. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **17**, 95–130 (1949). (Russian. Georgian summary)

The paper contains an approximate solution of Saint Venant's torsion and flexure problems for a slightly curved compound beam made up of materials with different Young's moduli, but with like Poisson's ratios. The cross-section S of the beam consists of an exterior simple closed contour L_0, containing within it a set of nonoverlapping regions S_i $(i=1, 2, \cdots, m)$ bounded by simple closed contours L_i. The contours L_i $(i=0, 1, \cdots, m)$ are given by equations of the form $f_i(x+kz^2, y)=0$, where the coordinate z is directed along the length of the beam, so that the axis of the beam, in the undeformed state, is a parabola whose curvature is determined by the small parameter k. The lateral surface of the beam is free of stress and the external forces act in the end $z=l$ of the beam.

The components of displacement are assumed to be continuous throughout the cross-section S. If the coordinates ξ, η, ζ defined by $\xi=x+kz^2$, $\eta=y$, $\zeta=z$ are introduced, the equations of contours L_i assume the forms $f_i(\xi, \eta)=0$. A change of variables in the elastostatic equations leads to a set of linear equations in the space of the variables ξ, η, ζ provided that one neglects terms of the order k^2. The paper contains solutions of such equations for the problems of extension, torsion, pure bending, and flexure by a transverse force, with illustrations of the first three of these problems. Illustrations deal with a slightly bent circular beam reinforced by a concentric circular core. The treatment follows N. I. Mushelišvili [Bull. Acad. Sci. URSS [Izvestiya Akad. Nauk SSSR] VII Sér. Cl. Sci. Math. Nat. **1932**, 907–945] and P. Riz [C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 110–113, 229–232 (1939); these Rev. **2**, 176].

I. S. Sokolnikoff (Los Angeles, Calif.).

Ōkubo, H. The torsion and stretching of spiral rods. I. Quart. Appl. Math. **9**, 263–272 (1951).

The torsion and stretching of a spiral rod is reduced to a two-dimensional problem by employing coordinates $x'+iy'=(x+iy)e^{ikz}$ where z is the axis of the helix and k is related to the slope of the spiral. Three plane harmonic polynomials in (x', y') are determined which provide a solution of the equilibrium equations for the displacements in the special case of an elliptic spiral section. A numerical example indicates that twisting is accompanied by axial displacement, and that axial loading induces twisting. The latter loading produces normal stresses Z_z which are nearly uniform over the section but attain their maximum values at the end of the minor axis of the ellipse, while the other normal stresses X_x' and Y_y' attain maximum values at the center of the section. For this case the maximum shearing stress X_z' occurs at the end of the minor axis.

D. L. Holl (Ames, Iowa).

Madejski, J. The torsion of prismatic bars with double-T cross-section. Arch. Méc. Appl., Gdańsk **3**, 61–87 (1951). (Polish. Russian summary)

Arutyunyan [Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 107–112 (1949); these Rev. **10**, 651] and Abramyan [ibid. **13**, 551–556 (1949); these Rev. **11**, 288] showed a method of solving the torsion problem for prismatic bars. The author solves a similar problem for a bar with an ideal double-T cross-section consisting of regular rectangles with sharp edges. The stress function U is given in the form of an infinite series depending on parameters L_k which are defined by an infinite system of simultaneous linear equations. The system, being regular, can be solved by successive approximations. The author obtains the formula for torsional rigidity, from which maximal stresses can also be evaluated. The results depend on T, a function of the cross-section dimensions. The values of T are tabulated. The numerical values of torsional rigidity found by the author are compared with corresponding values obtained from empirical formulas and they differ by -3.7% to $+319\%$. The empirical formulas appear to give only a reasonable agreement for very slender cross-sections. *T. Leser.*

Filonenko-Borodič, M. M. The problem of the equilibrium of an elastic parallelepiped with given loads on its boundaries. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 137–148 (1951). (Russian)

Using the variational method of Castigliano, Pankovich's stress tensor [Theory of Elasticity, Oborongiz, 1939], and the Maxwell stress function, the author solves the problem stated in the title. The stress function and the stresses are given as infinite series, where the terms are components of the Pankovich stress tensor, and the coefficients depend on the type of load. The author illustrates his solution with two examples: (1) a rectangular parallelepiped compressed by normal distributed loads on two opposite faces, the load distribution over the two faces being identical; (2) a rectangular parallelepiped compressed by normal distributed loads on two opposite faces, the load distribution over the two faces being different, the system being statically balanced. His method requires a laborious task of successive approximations. The author presents the first and the second approximation, showing that the first approximation is not accurate enough. The solutions seem to corroborate the Saint Venant principle, that the stresses approach a constant value throughout the cross-section area as the cross-section recedes from the load. The author claims that this shows the correctness of his method and its applicability in practice. *T. Leser* (Lexington, Ky.).

Grioli, Giuseppe. The structure of Airy's stress function in multiply connected regions. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1290**, 34 pp. (1951).

Translated from Giorn. Mat. Battaglini (4) **1**(**77**), 119–144 (1947); these Rev. **10**, 272.

O'Rourke, R. C. Three-dimensional photoelasticity. J. Appl. Phys. **22**, 872–878 (1951).

The author extends the classical method of Sommerfeld and Runge [Ann. Physik (4) **35**(**340**), 277–298 (1911), p. 290] in isotropic and homogeneous materials to media which are "weakly" anisotropic and inhomogeneous. The aim of the paper is threefold: (a) to generalize the usual eikonal equation of geometrical optics to anisotropic and inhomogeneous media and to discuss the Fermat principle associated therewith; (b) to apply this equation to the clarification of certain controversial procedures in three-dimensional photoelasticity; (c) to derive a method for treating spherically and axially symmetric problems in three-dimensional photoelasticity, with particular attention to residual stresses in quenched transparent materials such as glass. Experimental evidence is quoted to support the theoretical conclusions in (b) and (c) above.

The paper clarifies several controversial issues, such as the question of the effect of rotation of the secondary principal axes of the Fresnel ellipsoid on the interferometric patterns used in current photoelastic procedures. The results obtained in this connection agree with those of Goodman and Mindlin, derived in a different manner [L. Goodman and R. Mindlin, J. Appl. Phys. **20**, 89–95 (1949)], and show that the rotational effect mentioned previously is negligible in cases of practical interest. Another result is the development of a method for treating axially and spherically symmetric problems in photoelasticity, which extends earlier work by the author in this field [R. C. O'Rourke and A. W. Sáenz, Quart. Appl. Math. **8**, 303–311 (1950); these Rev. **12**, 303; A. W. Sáenz, J. Appl. Phys. **21**, 962–965 (1950)]. *A. W. Sáenz* (Washington, D. C.).

Tu, S. N. Stress analysis of cylindrical semi-monocoque structures. Eng. Rep. Nat. Tsing Hua Univ. **4**, no. 2, 50–77 (1950). (Chinese)

Analysis of semi-monocoque cylindrical structures is made for (1) closely spaced rigid bulkheads, (2) rigid bulkheads, with definite spacing and (3) elastic bulkheads at definite spacing. The rigid bulkhead is taken as being rigid only in its own plane. The displacement, the shear strain and hence the shear flow at different points can be expressed in terms of the displacement of a reference point in the bulkhead plane, a rotation about this point and the longitudinal displacement. From the stress-strain relations, the equilibrium conditions of the variation of stiffener load and the shear flows, the conditions that the shear flows balance the external forces and torsional moment, relations between the longitudinal displacements of different stiffeners are expressed as differential equations of second order. With end conditions of all stiffeners known, the longitudinal displacements and the stresses of different stiffeners are solved. For rigid bulkheads with definite spacing the shear flow in a panel is taken as constant. With m bulkheads and n stiffeners, there are $n(m-1)$ shear flows in $n(m-1)$ panels. With 3 equilibrium conditions on every vertical section between two bulkheads, there are $3(m-1)$ equilibrium conditions. The principle of minimum energy is used to obtain the additional equations. By methods of the calculus of variations, the shear flows in different panels are solved. For elastic bulkheads, the shear flow in the neighboring panels and external loads at this particular bulkhead are taken as loads on this bulkhead and the strain energy of the bulkhead is calculated accordingly. This strain energy of the different bulkheads is included in the minimum energy calculations. The above analysis can also be applied to structures with cut-out and torsional problems. Illustrative examples on cut-out and torsion are given for the above three types of bulkheads. *T. H. Lin* (Detroit, Mich.).

Wiegand, A. Die Berechnung der Grundschwingungszahlen von Spiralfedern. Z. Angew. Math. Mech. **31**, 35–46 (1951). (German. English, French, and Russian summaries)

The author considers a flat helical spring of the kind used in electrical measuring instruments and watches. He derives

approximate formulas for the first natural frequencies of vibrations in the plane of the spring and normal to it, using the Rayleigh-Ritz method. The derivation is based on the following assumptions and simplifications: the spring is a spiral of Archimedes; the amplitude of vibration is small as compared with the radius vector; tensile vibrations are negligible as compared with those due to bending; the spring has at least three turns. A numerical example shows that the natural frequency of vibrations normal to the plane of the spring is lower than the one in the plane of the spring. The natural frequencies should be as high as possible in order to avoid resonance; therefore, from the practical point of view, the evaluation of normal frequencies is more important. The author claims that the formulas gave a good agreement with a series of experimental results. *T. Leser.*

Matuzawa, Takeo. Elastische Wellen in einem anisotropen Medium. Bull. Earthquake Res. Inst. Tokyo **21**, 231–235 (1943). (German. Japanese summary)

Propagation of plane elastic waves in a hexagonal crystal is discussed and numerical results are given for several directions of propagation in beryllium. *I. S. Sokolnikoff.*

Sezawa, Katsutada, and Kanai, Kiyoshi. On the propagation of Rayleigh-waves in dispersive elastic media. Bull. Earthquake Res. Inst. Tokyo **19**, 549–553 (1941). (English. Japanese summary)

Si on complète arbitrairement les équations classiques de l'élasticité en introduisant une résistance de milieu proportionnelle au déplacement, les ondes longitudinales, les ondes transversales et les ondes de Rayleigh présentent toutes trois une dispersion dans laquelle le carré de la vitesse croît linéairement avec le carré de la longueur d'onde. Si on superpose deux milieux analogues, cette dispersion se combine, pour les ondes de Rayleigh, avec celle qui résulte de la stratification. Courbes de dispersion pour neuf cas.
J. Coulomb (Paris).

Sezawa, Katsutada, and Kanai, Kiyoshi. Transmission of arbitrary elastic waves from a spherical source, solved with operational calculus. I, II, III. Bull. Earthquake Res. Inst. Tokyo **19**, 151–161, 417–442 (1941); **20**, 1–19 (1942). (English. Japanese summary)

Operational methods are used to calculate the displacements in an infinite elastic solid due to prescribed normal and tangential stresses on a sphere $r=a$. The displacements can be expressed in terms of a pair of potentials Δ, $\bar{\omega}$ for dilatational and rotational waves, each of which satisfies a wave equation; the prescribed stresses are of the form $f(t)P_n(\cos\theta)$. Detailed calculations are made for $n=0$ and 2, and for various shapes of $f(t)$, the results being displayed on diagrams. *G. E. H. Reuter* (Manchester).

Press, Frank, and Ewing, Maurice. Theory of air-coupled flexural waves. J. Appl. Phys. **22**, 892–899 (1951).

The content of this paper appeared earlier in a report [the authors, Crary, Katz, and Oliver, Geophysical Research Papers No. 6, Air Force Cambridge Research Laboratories, Cambridge, Mass., 1950; these Rev. **12**, 880].

Edelman, F., and Drucker, D. C. Some extensions of elementary plasticity theory. J. Franklin Inst. **251**, 581–605 (1951).

A longer version of this paper appeared previously [Graduate Division of Applied Mathematics, Brown University, Providence, R. I., Tech. Rep. **A11–46** (1950); these Rev. **12**, 562]. *N. Coburn* (Ann Arbor, Mich.).

*****Drucker, D. C. Stress-strain relations in the plastic range. A survey of theory and experiment.** Graduate Division of Applied Mathematics, Brown University, Providence, R. I., Survey Rep. **A11–S1**, iv+308 pp. (1950).

This article contains an extensive review of the various known stress-strain relations for metals in the theory of plasticity. *N. Coburn* (Ann Arbor, Mich.).

Craggs, J. W. The influence of compressibility in elastic-plastic bending. Quart. J. Mech. Appl. Math. **4**, 241–247 (1951).

The bending of a beam by terminal couples is usually treated under the assumption that all components of stress except the longitudinal one are zero. The author points out that this assumption, in conjunction with the stress-strain law of Prandtl-Reuss leads to strain rates which, except for the case of an incompressible material, fail to satisfy the compatibility equations in the neighborhood of the elastic-plastic interface. To investigate the secondary stresses in this neighborhood, the author treats the following simplified problem: an infinite slab, bounded by the planes $y=\pm a$ and made of a perfectly plastic material for $x<0$ and a Prandtl-Reuss material for $x>0$, is subjected to uniform extension in the z direction. The solution is given in terms of powers of the small parameter $\alpha=\frac{1}{2}-\nu$, where ν is Poisson's ratio in the elastic range. It is found that, in the neighborhood of the plane $x=0$, the secondary stresses are of the order αY, where Y is the yield stress in simple tension.
W. Prager (Providence, R. I.).

Hill, R., and Siebel, M. P. L. On combined bending and twisting of thin tubes in the plastic range. Philos. Mag. (7) **42**, 722–733 (1951).

The combined elastic-plastic bending and twisting of a cylindrical tube of small but uniform wall thickness and otherwise arbitrary cross section is investigated under the assumptions that the tube is made of a Prandtl-Reuss material, and that the loading program avoids unloading of any plastically strained element of the tube. Formulas for the stresses, warping, and limiting values of the bending and twisting couples are obtained. They are evaluated numerically for various loading programs in the case of a circular cross section. Finally, a mathematically much simpler treatment based on the Hencky stress-strain relations (which are admittedly not acceptable physically) is presented. This is found to furnish results close to those of the preceding theory whenever the ratio of the angles of bend and twist remains constant during the loading process.
W. Prager (Providence, R. I.).

Lepik, Yu. R. Two remarks on the theory of stability of plates beyond the elastic limit, taking account of the compressibility of the material. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 553–557 (1950). (Russian)

A. A. Ilyushin has discussed the problem for incompressible materials in terms of the theory of small elastic-plastic deformations. The present paper discusses the validity of Ilyushin's results for compressible materials. It is shown that the solution under the assumption of incompressibility approaches the true solution as the difference between the yield stress and the buckling stress increases. On the other hand, the solution based on the assumption of small plastic deformation can be used only so long as the buckling stress does not exceed the yield stress by more than 3 per cent. The second part of the paper formulates four conditions

under which the relative thickness of the plastic zone remains constant in compressible materials. It is shown that the assumption of incompressibility leads to an overestimate of the relative thickness. *H. I. Ansoff.*

Popov, S. M. On the cylindrical buckling of plates beyond the elastic limit. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 543–552 (1950). (Russian)

The paper presents an exact formulation of the cylindrical buckling of a long, thin, rectangular plate under uniform compression. A. A. Ilyushin has determined approximate lower and upper bounds of the relative thickness of the plastic layer in the middle of the plate and concluded that the unloading buckling load will differ little from the value obtained from the approximate solution. The present paper shows, that although Ilyushin's conclusions were not justified on this basis, the approximate solution does indeed yield a good approximation to the critical load. The author also determines the boundary between the plastic and the elastic-plastic zones of the plate, as well as the relative thickness of the plastic zone in the middle of the plate.

H. I. Ansoff (Santa Monica, Calif.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Mertens, Robert. On the theory of the diffraction of light by supersonic waves. Simon Stevin 27, 212–230 (1950).

A beam of monochromatic light passes through a column of liquid disturbed by a transverse beam of ultrasonic frequency, the directions of propagation being the z and x axes, respectively. If $\Phi(x, z, t) \exp(ikct)$ is the electric field of the light wave, the equation to be solved is

$$\text{(A)} \qquad \Phi_{xx}+\Phi_{zz}+k^2\mu^2(x,t)\Phi=0$$

where μ, the refractive index, exhibits a harmonic dependence on x and t, small compared to the undisturbed value, due to the ultrasonic disturbance. Posing a harmonic dependence in z, the author reduces (A) to Mathieu's equation and obtains a solution which is in good accord with experiment and therefore, according to the author, superior to the earlier analysis of L. Brillouin [La diffraction de la lumière . . . , Actualités Sci. Ind., no. 59, Hermann, Paris, 1933, pp. 5–6, 21–22]. *J. W. Miles* (Auckland).

Mertens, Robert. On the theory of the diffraction of light by supersonic waves. II. Simon Stevin 28, 1–12 (1951).

The author extends the analysis of the paper reviewed above, which dealt with progressive, ultrasonic waves, to deal with interference by standing, ultrasonic waves.

J. W. Miles (Auckland).

Gordon, A. N. The field induced by an oscillating magnetic dipole outside a semi-infinite conductor. Quart. J. Mech. Appl. Math. 4, 106–115 (1951).

Formulas are given for the external field produced by an oscillating magnetic dipole located at the point $(0, 0, h)$ outside a semi infinite conductor of unit permeability and conductivity k. The moment M of the dipole varies sinusoidally with period $2\pi/p$. The induced potential in a point P with cylindrical coordinates (ρ, φ, z) is

$$\text{(1)} \qquad \Omega_i=\alpha^2 M\int_0^\infty \frac{(t^2+i)^{\frac{1}{2}}-t}{(t^2+i)^{\frac{1}{2}}+t}e^{-tz}J_0(tr)t\,dt,$$

where $\alpha^2=4\pi kp/c^2$, $r=\alpha\rho$, $z=(h+\zeta)\alpha$. Hence the magnetic field components at the surface $\zeta=0$ in case the dipole is at the surface $(h=0)$ are found to be

$$\text{(2)} \qquad H_z=\alpha^3 M\int_0^\infty \frac{(t^2+i)^{\frac{1}{2}}-t}{(t^2+i)^{\frac{1}{2}}+t}J_0(tr)t^2\,dt,$$

$$\text{(3)} \qquad H_\rho=\alpha^3 M\int_0^\infty \frac{(t^2+i)^{\frac{1}{2}}-t}{(t^2+i)^{\frac{1}{2}}+t}J_1(tr)t^2\,dt.$$

The integrals (2) and (3) can be evaluated in terms of modified Bessel functions with the argument $r\sqrt{i}$. For the integral (1) a series in ascending powers of R $(R^2=\zeta^2+(h-z)^2)$ is given, suitable for the numerical computation for values of R not greater than 2. For values of R above 10, (1) is represented as a series in descending powers of R. The same component as in (2) in the case of a circular wire of radius d which carries an alternating current cI is given by a similar integral. The circuit has its center at $(0, 0, h)$ and its plane is parallel to the bounding face of the conductor. For this integral a series suitable for computations involving small values of r and $a\equiv\alpha d$ is given. *F. Oberhettinger.*

Hart, Robert W., and Montroll, Elliott W. On the scattering of plane waves by soft obstacles. I. Spherical obstacles. J. Appl. Phys. 22, 376–386 (1951).

The scattering from a sphere whose index of refraction is close to that of the surrounding medium is treated for both the acoustical and electromagnetic cases. The exact formulas for the scattered and transmitted waves are obtained in terms of an infinite series which the authors show may be summed approximately if the assumption stated above is employed. The results are compared with exact calculations as well as the approximate results of Rayleigh and Gans valid for long wavelengths. *H. Feshbach.*

Roždestvenskiĭ, B. L., and Četaev, D. N. On the suppression of reflections in wave guides with variable cross-section. Doklady Akad. Nauk SSSR (N.S.) 79, 427–430 (1951). (Russian)

Kahan, Th., und Eckart, G. Über die Ausbreitung elektromagnetischer Wellen in einem atmosphärischen Wellenleiter. Z. Naturforschung 5a, 334–342 (1950).

The authors consider the propagation of electromagnetic waves emitted by a vertical magnetic dipole located above a perfectly conducting flat earth. The atmosphere suffers a discontinuity in dielectric constant at a height h above the earth. The solutions satisfying the boundary conditions are in the usual form suggested by the integral representation of e^{ikR}/R in cylindrical coordinates. The far field values are obtained by saddle point integration. An expansion in which the first terms are the point source and its image with respect to the earth is also obtained. The paper concludes with a discussion of the physical consequences of the formulas developed including, for example, the existence of shadow regions. *H. Feshbach* (Cambridge, Mass.).

Bedini, Lidia. Sulla distribuzione della corrente alternata in un sistema di conduttori cilindrici paralleli. Rivista Mat. Univ. Parma 1, 425–431 (1950).

The distribution of alternating current in a system of parallel conductors is reduced to a system of integral equations or to an equivalent system of differential equations and the uniqueness of the solution is proved.

I. Opatowski (Chicago, Ill.).

Méring, J. L'interférence des rayons X dans les systèmes à stratification désordonnée. Acta Cryst. 2, 371–377 (1949).

The scattering of X-rays by parallel layers in disordered stacking is calculated, taking account of the number of layers per stack. The result is a more general formula than that obtained by Hendricks and Teller [J. Chem. Phys. 10, 147–167 (1942)] for random stacks of an infinite number of layers. Systems with complete or partial randomness are discussed. The latter case comprises crystals with "mistakes". The influence of the nature of the mistakes (accidental shifts or twinning along certain planes) is studied as is also the influence of their distribution. (Author's summary.)

W. Nowacki (Bern).

Moussa, André, et Lafoucrière, Joseph. Sur les expressions analytiques du potentiel-vecteur d'un champ magnétique à symétrie de révolution. C. R. Acad. Sci. Paris 233, 139–141 (1951).

Strutt, M. Ueber die Berechnung des elektrostatischen Feldes moderner Elektronenröhren. Schweiz. Bauztg. 67, 36–37 (1949).

Smythe, W. R. Electric and magnetic forces between sphere and wire. J. Appl. Phys. 22, 521–522 (1951).

Formulas are derived for the force between (a) a dielectric sphere and a charged wire and (b) a permeable sphere and a circular cylinder carrying current i. The results simplify if in case (a) the sphere is an ideal conductor and in case (b) the permeability of the sphere is large. Then the electric force in mks units is respectively

$$F_e = q^2a^3(\pi\epsilon_v b)^{-1}(b^2-a^2)^{-\frac{1}{2}}\sin^{-1}(a/b),$$

$$F_m = \mu_v i^2a^2(\pi b^{-1})(b^2-a^2)^{-\frac{1}{2}}\sin^{-1}(a/b)$$

where q is the charge per unit length on the wire, a the sphere radius and b its distance from its center to the wire.

F. Oberhettinger (Washington, D. C.).

Gordon, A. N. Electromagnetic induction in a uniform semi-infinite conductor. Quart. J. Mech. Appl. Math. 4, 116–128 (1951).

Two independent solutions of Maxwell's equation for the magnetic field B inside a uniform sourcefree medium of conductivity μ and permeability κ, when displacement currents are neglected, are curl (kW) and grad $(\partial W/\partial z)-\alpha^{-1}\partial W/\partial t$, where $\alpha=c^2/4\pi\mu\kappa$, k is the unit vector along the z-axis, and W is a solution of the scalar equation $\nabla^2W=\alpha^{-1}\partial W/\partial t$. Applications are made to the problem of the induction of currents in a uniform semi infinite conductor by external magnetic or electric fields. The following case is investigated: The region $z>0$ is occupied by conducting material while an external magnetic inducing system is confined to the region $z<-h$, $h>0$, attention being called to the analogy with a corresponding problem in heat conduction.

F. Oberhettinger (Washington, D. C.).

Friedman, Bernard. Report on a conference on dynamics of ionized media. New York University, Washington Square College, Research Group, Research Rep. No. EM-30, i+iii+22 pp. (1951).

The conference was held at University College, University of London, March 19–21, 1951. The topics discussed were chiefly magneto-hydrodynamics and plasma oscillations.

★Ivanenko, D., and Sokolov, A. Klassičeskaya teoriya polya (novye problemy). [**The Classical Theory of Fields (New Problems)**]. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949. 432 pp.

This comprehensive work is directed to research workers and post-graduate students in physics. The titles and lengths of its five chapters are as follows: (I) General theory of the δ-function (27 pp.); (II) Static equations of elliptic type (36 pp.); (III) Time-dependent equations (44 pp.); (IV) Classical electrodynamics (170 pp.); (V) Classical mesodynamics (150 pp.). The word "classical" is used to exclude quantum theory but not special relativity.

The first three chapters give a thorough discussion of the δ and Δ functions and their use in constructing Green's functions for the solution of linear partial differential equations including Klein's equation, the n-dimensional wave equation, and the damped wave equation in three dimensions. These chapters constitute a clear and systematic introduction to mathematical methods now widely used in the quantum theory of fields.

The fourth chapter applies the above mathematical methods to selected advanced topics in electrodynamics most of which concern the nature or motion of a single electron. The power of the δ-function in the hands of the authors to systematize and simplify this subject is most striking. The chapter includes a discussion of Čerenkov's theory of electrons moving in a medium with speed greater than that of light in the medium (the "super-light-signal electron"); of Born and Infeld's nonlinear electrodynamics; of the theory of Bopp and others of electromagnetic fields obeying field equations of order higher than two; of the λ-process; of the theory of the betatron including Schott's theory of radiation damping.

The final chapter gives a fairly standard treatment of the scalar, pseudoscalar, vector and pseudovector meson fields up to the point of quantization. The last section of the book briefly compares the gravitational field of General Relativity with electromagnetic and meson fields.

In the opinion of the reviewer, the authors have succeeded admirably in weaving a wealth of information from much literature of the past fifteen years into a thoroughly readable introduction to the mathematical methods and results of "classical" electrodynamics and meson theory.

A. J. Coleman (Toronto, Ont.).

Green, H. S., and Cheng, K. C. The reciprocity theory of electrodynamics. Proc. Roy. Soc. Edinburgh. Sect. A. 63, 105–138 (1951).

The authors formulate a reciprocally invariant Lagrangean function for a system of electrons in interaction with the electromagnetic field along the lines laid down in a previous paper [Born, Green, Cheng, and Rodriguez, same Proc. 62, 470–488 (1949); these Rev. 11, 147]. The field equations obtained from this Lagrangean are solved for the case of the unaccelerated motion of an electron and for the case of an electron in arbitrary motion. The results of the earlier paper are used to derive a Hamiltonian for the electron and field and this enables a quantized formulation of the theory from which some of the usual divergences are absent.

A. H. Taub (Urbana, Ill.).

Darlington, Sidney. The potential analogue method of network synthesis. Bell System Tech. J. 30, 315–365 (1951).

This paper deals with the application of potential theory to the problem of finding a network whose gain or phase characteristic over a frequency band is to approximate to

within a given tolerance a prescribed characteristic. The problem is regarded as solved when the zeros and poles of the desired network function are located. Since the gain and phase of linear networks are correspondingly the real and imaginary parts of an analytic function, they may be associated with the potential and stream functions of charge distributions in a plane, and the charge locations correspond to the locations of the zeros and poles of the network function. Hence when the network characteristic is prescribed, a solution may be afforded when a charge distribution is found having the required potential or stream function. Since such distributions are generally continuous, a lumped charge approximation is necessary. The constraints on admissible solutions imposed by the conditions of physical realizability are discussed together with reasons for a suitable choice of contours on which the charge is to be located. Before treating the general case, simple applications of the potential analogue method to condenser delay and filter networks is described. The general potential solution given by Green's formula is unsuitable for network synthesis since it requires double layer distributions; hence the author is chiefly concerned with obtaining solutions which yield potential distributions continuous across the contour produced by a single layer of charge. Some methods for finding the analytic continuation of the potential across the unit circle are given as well as suitable transformations of general contours into the unit circle. *R. Kahal.*

Kirschner, Ulrich. Darstellung einer allgemeinen Röhrenschaltung durch eine Kettenmatrix. Arch. Elektr. Übertragung **5**, 190–196 (1951).

The transmission matrix of electron tube circuits is developed, from which equivalent circuit representations in a form convenient for computations may be derived. This is done in a very general way both for triodes and pentodes taking into account all interelectrode capacitances and cathode circuit impedances. *R. Kahal* (St. Louis, Mo.).

Tellegen, B. D. H., and Klauss, E. Resonant circuits coupled by a passive four-pole that may violate the reciprocity relation. Philips Research Rep. **6**, 86–95 (1951).

The design equations for a two-terminal pair network which is to provide coupling between two resonant circuits are given. It is desired to obtain the maximum overall transfer at a single frequency when the shape of the resonance curve is prescribed. By employing a new network element, the gyrator, unfamiliar to this reviewer, a gain in the maximum transfer of $(1+\sqrt{2})/\sqrt{2}$ over the optimum coupling network using bilateral R, L, and C elements is obtained. *R. Kahal* (St. Louis, Mo.).

Quantum Mechanics

Falk, Gottfried. Axiomatik als Methode physikalischer Theorienbildung. Z. Physik **130**, 51–68 (1951).

Author investigates the relation between classical and quantum mechanics by formulating these theories by systems of axioms having a common part (in fact, by systems that are almost the same). A formal relation is established between Maxwell's equations and Dirac's relativistic quantum theory, from which it is concluded that restmass must be regarded as an operator that cannot always be diagonalized simultaneously with the total energy operator. The methodology of this paper is to regard the well-known relations among the Poisson brackets $[\varphi, \psi]$ as purely formal axioms defining an operation [ ] over the ring of formal power series φ, ψ in $2n$ commuting quantities q_i, p_i. This operation [ ] defines a family of one-parameter transformations representing motions in the manner of classical mechanics. An exactly parallel treatment, except for the condition $p_iq_k-q_kp_i=(h/i)\delta_{ik}$, leads to a formulation of quantum mechanics. It is indicated that no other mechanical system can be obtained by the formal operation [ ]. The Heisenberg, Schrödinger, and mixed (Schwinger) representations are obtained very simply from the abstract formulation. A relation between the Maxwell and Dirac theories is also obtained by a similar formal power series definition of an operation [ ]. *C. C. Torrance* (Annapolis, Md.).

Géhéniau, J., et Demeur, M. Solutions singulières des équations de Dirac, tenant compte d'un champ magnétique extérieur. Physica **17**, 71–75 (1951).

The authors determined without approximation a matrix $S_{\alpha\beta}(P, P')$ $(\alpha, \beta=1, \cdots, 4)$ whose elements are functions of two points in space-time such that for fixed β and P', $S_{\alpha\beta}(P, P')$ satisfies the Dirac equation for an electron in a uniform and constant magnetic field and for fixed α and P, $S_{\alpha\beta}$ satisfies the adjoint equation. *A. H. Taub.*

Koppe, H. Die Streuung eines Teilchens an einer Potentialschwelle. Z. Naturforschung **6a**, 229–233 (1951).

A new method of handling scattering problems is developed and the scattering of a wave packet by different potential barriers is considered. The method is to first take the Laplace transform with respect to time of the Schrödinger equation; the resulting inhomogeneous equation is solved with the aid of an appropriate Green's function; the inverse transform is then evaluated approximately for the case of a sharply defined momentum. Finally, reflection and transmission coefficients are found. *T. E. Hull.*

Dalitz, R. H. On higher Born approximations in potential scattering. Proc. Roy. Soc. London. Ser. A. **206**, 509–520 (1951).

This paper contains a critique of previous attempts to calculate second and higher Born approximations. The correct procedure is outlined and illustrated for (1) relativistic Coulomb scattering (2nd Born approximation) (2) nonrelativistic Coulomb scattering (3rd Born approximation) (3) relativistic Coulomb scattering of spin 1 particles. In the second case the author was able to show that the higher Born approximations only change the phase of the first Born approximation. *H. Feshbach* (Cambridge, Mass.).

Kato, Tosio. On the convergence of the perturbation method. II, ~~I~~. Progress Theoret. Physics **5**, ~~207–212~~ 95–101, ~~(1950).~~ 207-212, (1950)

The author considers the possibility that the perturbation theory, even when it does not converge, may provide an asymptotic series. For this purpose he makes use of the well-known fact that the even order perturbation theory may be obtained from a variational principle. It is, of course, necessary that the function developed by the perturbing operator acting on the eigenfunctions exist. An example where this condition is not met is given, and yet each term in the formal perturbation series is finite. *H. Feshbach.*

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